

Time-splitting for the random nonlinear Schrödinger equation

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- 1 Introduction
- 2 The nonlinear Schrödinger equation with random dispersion
- 3 Time-splitting scheme for NLS with random dispersion

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The equation

We aim to study the following NLS equation :

NLS with random dispersion

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = i \mathring{W}(t) \frac{\partial^2 u}{\partial x^2}(t, x) + g(u(t, x)) & \forall (t, x) \in (0, 1] \times \mathbb{R} \\ u(t = 0, x) = u_0(x) \end{cases}$$

where

- u is unknown and u_0 is the initial condition,
- g is a nonlinear function,
- \mathring{W} is a white noise (i.e. the "derivative" of a Brownian motion W) :

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Roughly speaking, $\overset{\circ}{W}$ can be seen as a stationary Gaussian process satisfying for every $t, s \in [0, 1]$,

$$\mathbb{E}(\overset{\circ}{W}(t)) = 0 \quad \& \quad \mathbb{E}(\overset{\circ}{W}(s) \overset{\circ}{W}(t)) = \delta(t - s).$$

It corresponds to the normalized independent infinitesimal increments of a **Brownian motion** W , which is a Gaussian process satisfying for every $t, s \in [0, 1]$,

$$\mathbb{E}(W(t)) = 0 \quad \& \quad \mathbb{E}(W(s)W(t)) = \min(t, s).$$

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Sample path of the white noise \dot{W}

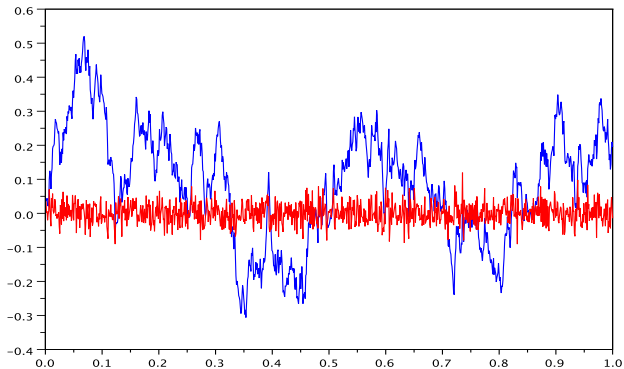


FIGURE: Blue : W ; Red : \dot{W} .

Origin : nonlinear optic

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Propagation in optical fiber

$$\begin{cases} \frac{\partial u}{\partial z}(z, \tau) = i \overset{\circ}{W}(z) \frac{\partial^2 u}{\partial \tau^2}(z, \tau) + g(u(z, \tau)) & \forall (z, \tau) \in (0, 1] \times \mathbb{R} \\ u(z = 0, \tau) = u_0(\tau) \end{cases}$$

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- z is the position on the fiber, τ is the time,
- $u(z, \tau)$ represents the electric field on the position z on the fiber,
- $\overset{\circ}{W}$ is the white noise dispersion.



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NLS with random dispersion

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is solved by following the procedure :

- $v_0(x) \xrightarrow{FT} \widehat{v}_0(\xi)$
- we solve the Fourier linear equation $\frac{\partial \widehat{v}}{\partial t}(t, \xi) = -i \mathring{W}(t) \xi^2 \widehat{v}(t, \xi)$, which gives $\widehat{v}(t, \xi) = \widehat{v}_0(\xi) \exp(-i \xi^2 (W(t) - W(t_0)))$ (Itô-Stratonovich calculus)
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We define X by

$$X(t_0, t)v_0(x) = v(t, x).$$

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is written in the integral form :

NLS with random dispersion in integral form

$$u(t, x) = X(0, t)u_0(x) + \int_0^t X(\theta, t)g(u(\theta, x))d\theta \quad \forall (t, x) \in (0, 1] \times \mathbb{R}$$

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Existence and uniqueness

Theorem

Let $u_0 \in L_x^2$. We assume that g is Lipschitz. Then, there exists a unique solution $u \in \mathcal{C}([0, 1], L_x^2)$ a. s. Moreover, if $u_0 \in H_x^2$, g two times differentiable and its derivatives up the order 2 are bounded, then $u \in \mathcal{C}([0, 1], H_x^2)$ a. e.

The proof consists of a fixed point procedure on the integral form

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Extensions

For a cubic nonlinearity $g(u) = i|u|^2u$



A. De Bouard and A. Debussche,

The nonlinear Schrödinger equation with white noise dispersion,
Journal of Functional Analysis, 259, pp. 1300-1321 (2010)

For a quintic nonlinearity $g(u) = i|u|^4u$



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Notation

We introduce three operators :

- S such that $v(t, x) = S(t_0, t)v_0(x)$ solves

$$\frac{\partial v}{\partial t} = i \overset{\circ}{W} \frac{\partial^2 v}{\partial x^2} + g(v) \quad \text{and} \quad v(t = t_0, x) = v_0(x)$$

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The scheme : definition and convergence

We subdivide $[0, 1]$ into N intervals of length $h = 1/N$ and we aim to construct $\{u_n^h\}_{n \in \{0, 1, \dots, N\}}$ such that $\{u_n^h\}_{n \in \{0, 1, \dots, N\}} \approx_{h \rightarrow 0} \{u(nh, \cdot)\}_{n \in \{0, 1, \dots, N\}}$. We let :

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Let $u_0 \in H^2(\mathbb{R}, \mathbb{C})$ and assume that g is sufficiently regular. Then there exists C such that for every $h \in]0, 1]$ and n satisfying $nh \leq 1$, we have

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(Remember that $u(nh, \cdot) = S((n-1)h, nh)u((n-1)h, \cdot) \dots$)

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Simulations

The order of convergence is bounded below by 1/2.

We aim to illustrate the previous result by simulation.

We let $\varepsilon(h) = \|u_N^h - u(1, \cdot)\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))}$ for every $h = 2^{-m}$ where m is integer.
The order of convergence will be given by

$$\alpha(m) = \log_2 \left(\frac{\varepsilon(2^{-m})}{\varepsilon(2^{-m-1})} \right).$$

Indeed : assume that $\varepsilon(h) \approx h^{\alpha^*}$ for $h \rightarrow 0$. Then

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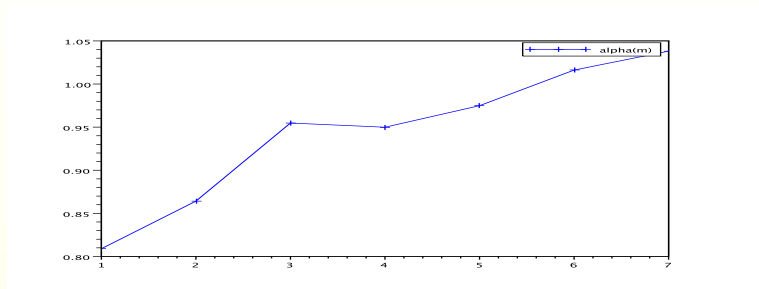


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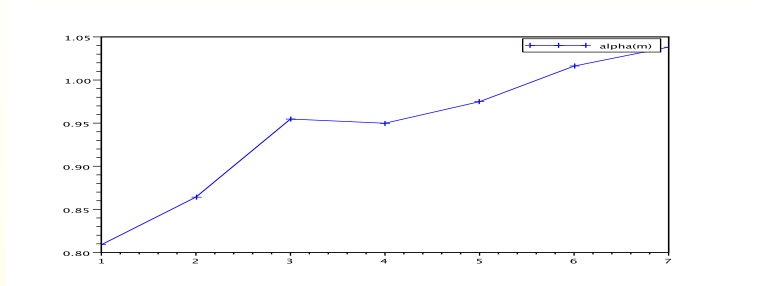


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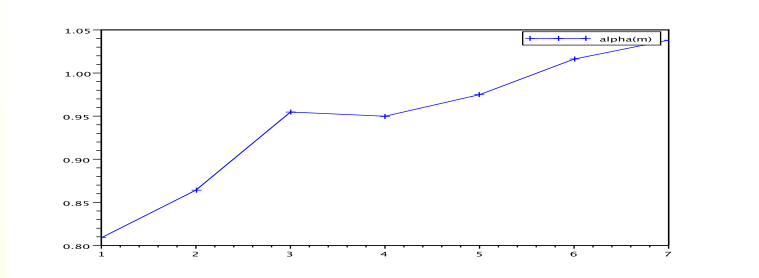


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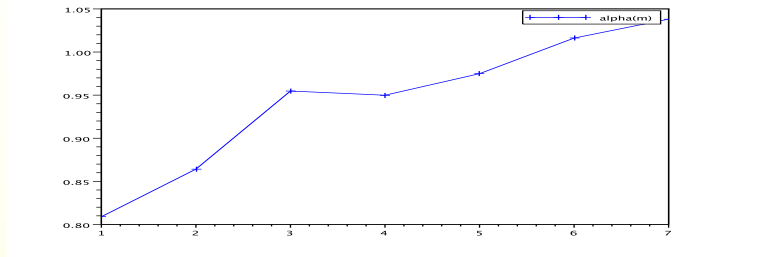


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Previously we established an estimate on the local error :

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In order to study more precisely the scheme we establish :

Theorem (M. (preprint 2011))

We assume that $v \in H^4$ and g sufficiently regular. Then for every $t_0 \leq t \in [0, 1]$,

$$\|\mathcal{L}[t_0, t, v] - \mathcal{G}(v)\mathcal{I}(t_0, t)\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))} \leq C(t - t_0)^2$$

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Previously we established an estimate on the local error :

$$\mathcal{L}[t_0, t, v] = \|S(t_0, t)v - Y(t - t_0)X(t_0, t)v\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))} \leq C(t - t_0)^{3/2}.$$

In order to study more precisely the scheme we establish :

Theorem (M. (preprint 2011))

We assume that $v \in H^4$ and g sufficiently regular. Then for every $t_0 \leq t \in [0, 1]$,

$$\|\mathcal{L}[t_0, t, v] - \mathcal{G}(v)\mathcal{I}(t_0, t)\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))} \leq C(t - t_0)^2$$

with $\mathcal{G}(v) = i \frac{\partial^2 g(v)}{\partial x^2} - \left\langle (\nabla g)(v), i \frac{\partial^2 v}{\partial x^2} \right\rangle$ and $\mathcal{I}(t_0, t) = \int_{t_0}^t (W_r - W_\theta) d\theta.$

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$$\|u_n^h - u(nh, \cdot)\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))}^2 = \mathbb{E}[\|u_n^h - u(nh, \cdot)\|_{L^2}^2] \leq Ch^2.$$

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Applying the norm $\|\cdot\|_{L^2}$, we get

$$\|u_n^h - u(nh, \cdot)\|_{L^2}^2 = \mathcal{E}_1(n, h) + 2\mathcal{R}\mathcal{E}_2(n, h)$$

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We first have (M. (2006), thanks to $\mathcal{L}[t_0, t, v_0]^2 \leq C(\|v_0\|_{H^2})(t - t_0)^3$)

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$$\mathbb{E}[\langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2}] \approx (1+h)^{n-k} h^2 (1+h)^{k-1-j} h^2 (1+h)^{j-1} \leq Ch^4$$

Exact order of convergence

We successively make the approximations

- $Z_l^h v \approx v + iW_l^h \partial_x^2 v$, $(W_l^h = (W_{lh} - W_{(l-1)h}))$
- $\mathcal{L}[(l-1)h, lh, v] \approx \mathcal{G}(v)\mathcal{I}_l^h$, $(\mathcal{I}_l^h = \mathcal{I}((l-1)h, lh))$
- $S_l^h v \approx v + iW_l^h \partial_x^2 v$,

in $\langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2}$ with $E_j^{(n,h)} = Z_n^h \cdots Z_j^h S_{j-1}^h \cdots S_1^h u_0 - Z_n^h \cdots Z_{j+1}^h S_j^h \cdots S_1^h u_0$

Very roughly, something "like this" will appear :

$$\begin{aligned} \langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2} \approx & |Id + iW_n^h|^2 \cdots |Id + iW_{k+1}^h|^2 (Id + iW_k^h) \mathcal{I}_k^h |Id + iW_{k-1}^h|^2 \cdots \\ & \cdots |Id + iW_{j+1}^h|^2 (Id - iW_j^h) \mathcal{I}_j^h |Id + iW_{j-1}^h|^2 \cdots |Id + iW_1^h|^2 \end{aligned}$$

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Exact order of convergence

Finally, we get

$$\begin{aligned} \mathbb{E}(\mathcal{E}_2(n, h)) &= \mathbb{E} \left(\sum_{j,k=1, j < k}^n \langle E_j^{(n,h)}, E_k^{(n,h)} \rangle_{L^2} \right) \\ &\leq C \sum_{j,k=1, j < k}^n h^4 \leq Ch^2, \end{aligned}$$

then

$$\|u_n^h - u(nh, \cdot)\|_{L^2(\Omega, L^2(\mathbb{R}, \mathbb{C}))}^2 = \mathbb{E}[\|u_n^h - u(nh, \cdot)\|_{L^2}^2] \leq Ch^2.$$

This concludes the "proof".

Thank you !

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