Transparent boundary conditions for locally perturbed infinite hexagonal periodic media

I. Lacroix-Violet

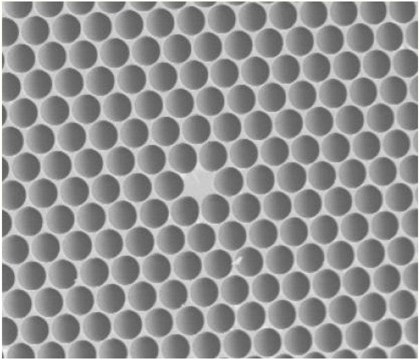
Joint work with C. Besse, J. Coatléven, S. Fliss and K. Ramdani

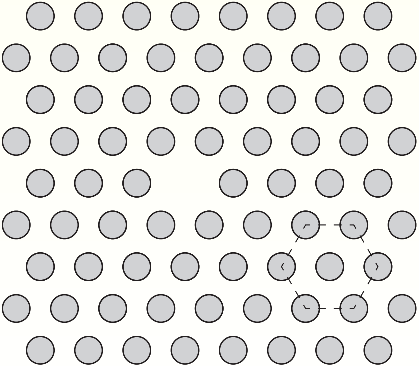
Laboratoire Paul Painlevé
INRIA Lille Nord-Europe - SIMPAF

Goal

☛ Determination of the DtN operator for infinite, lossy and locally perturbed hexagonal periodic media.

Photonic crystal





Overview of the method

- Factorization of the DtN operator involving two non local operators
  - a DtN operator for a half-space problem
  - a DtD operator taking advantage of the symmetry properties

Overview of the method

- **Factorization of the DtN operator involving two non local operators**
  - **a DtN operator for a half-space problem**
  - **a DtD operator taking advantage of the symmetry properties**

- **Characterization of the DtN operator for a half-space problem**
  - ➢ Floquet-Bloch transform
  - ➢ Family of elementary strip problems
  - ➢ Family of stationary Riccati equations

Overview of the method

- **Factorization of the DtN operator involving two non local operators**
  - **a DtN operator for a half-space problem**
  - **a DtD operator taking advantage of the symmetry properties**

- **Characterization of the DtN operator for a half-space problem**
  - Floquet-Bloch transform
  - Family of elementary strip problems
  - Family of stationary Riccati equations

- **Characterization of the DtD operator**
  - Affine valued equation $\rightarrow$ non standard integral equation

Outlines

Introduction

Previous works

Two classes of methods

☛ The periodicity can be handled via homogenization techniques

- G. Allaire, C. Conca, M. Vanninathan (1999),
- G. Bouchitté, S. Guenneau, F. Zolla (2010).

☛ Keeping the periodicity but considering only

➢ finite media

- M. Ehrhardt, H. Han, C. Zheng (2009),
- M. Ehrhardt, C. Zheng (2010),
- Z. Hu, Y. Lu (2008),
- L. Yuan, Y.-Y. Lu (2006, 2007).

➢ media that can be reduced to finite domains

- A. Figotin, A. Klein (1997, 1998),
- R.-C. Gauthier (2007).

Previous works

☛ Infinite two dimensional periodic media containing local defects

- ❑ S. Fliss (PhD Thesis, 2009);
- ❑ S. Fliss, P. Joly (2009).

Main assumptions of these works

- ➢ orthogonality of directions of periodicity,
- ➢ commensurate periodicity lengths,
- ➢ dissipative Helmholtz equation.

For hexagonal periodic media, the corresponding periods would not be commensurate

Applications of hexagonal lattices

☛ Quantum mechanics: the mathematical formulation leads to Schrödinger operator

$$Au := -\Delta u + (V + ip)u.$$

☛ Phononics: the operator involved is the elasticity system

$$Au := -\text{div}\sigma(u) + \omega^2 \rho u$$

☛ Photonics: electromagnetic propagation is described by the vector Maxwell's equations which in 2D reduce to

Transverse electric polarizations case: $Au := \Delta u + \omega^2 n^2 u$

Transverse magnetic polarizations case: $Au := -\text{div}\left(\frac{1}{n^2}\nabla u\right) + \omega^2 u$

Applications of hexagonal lattices

☛ Quantum mechanics: the mathematical formulation leads to Schrödinger operator

$$Au := -\Delta u + (V + ip)u.$$

☛ Phononics: the operator involved is the elasticity system

$$Au := -\text{div}\sigma(u) + \omega^2 \rho u$$

☛ Photonics: electromagnetic propagation is described by the vector Maxwell's equations which in 2D reduce to

Transverse electric polarizations case: $Au := \Delta u + \omega^2 n^2 u$

Transverse magnetic polarizations case: $Au := -\text{div}\left(\frac{1}{n^2}\nabla u\right) + \omega^2 u$

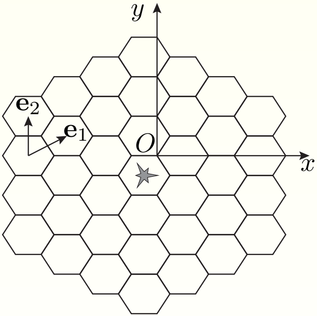
The defect is taken into account by adding a bounded obstacle
or locally perturbing the coefficients

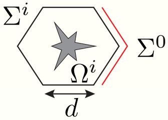
Problem setting

☛ Domain: Infinite photonic crystal $\Omega = \mathbb{R}^2$ with a localized defect

Main notations

- Ω^i: cell with defect
- Σ^i: boundary of Ω^i
- $\Omega^e = \Omega \setminus \overline{\Omega^i}$
- $\mathbf{e}_1$, $\mathbf{e}_2$: directions of periodicity





☛ Model problem: Dissipative Helmholtz equation

$$\Delta u + \rho u = f, \quad \text{in} \quad \Omega.$$

slide 14

Problem setting

☛ Hexagonal symmetry

Definition for a domain

A domain $\mathcal{O}$ of $\mathbb{R}^2$ has hexagonal symmetry if there exists a rotation of angle $2\pi/3$, denoted $\Theta_{2\pi/3}$ under which $\mathcal{O}$ is invariant.

Definition for a function

Let $\mathcal{O}$ be an open set with hexagonal symmetry and let g be a real or complex valued function defined on $\mathcal{O}$. Then, g has hexagonal symmetry if

$$g = g \circ \Theta_{2\pi/3}.$$

Problem setting

☛ Assumptions

- ρ is a local perturbation of a hexagonal periodic function ρ_{per}

$$\rho = \rho_{\text{per}} + \rho_0,$$

  - ✓ for all $\mathbf{x} = (x, y) \in \Omega$ and all $(p, q) \in \mathbb{Z}^2$, $\rho_{\text{per}}(\mathbf{x} + p\mathbf{e}_1 + q\mathbf{e}_2) = \rho_{\text{per}}(\mathbf{x})$
  - ✓ ρ_{per} and ρ_0 have hexagonal symmetry
  - ✓ $\text{Supp}(\rho_0) \subset \Omega^i$

- ρ satisfies the dissipation property

$$|\text{Im}\,\rho(\mathbf{x})| \geq \rho_b > 0, \qquad \forall \mathbf{x} \in \Omega. \tag{1}$$

- The source f is compactly supported in Ω^i and has hexagonal symmetry.

(1) guarantees existence and uniqueness of finite energy solutions

☛ Goal: Propose a method to solve the Helmholtz equation in the infinite domain Ω under these assumptions

Idea of the method

☛ Key idea Reduce the problem to a boundary value problem set in the cell Ω^i

$\Longrightarrow$ Derive suitable transparent boundary condition on Σ^i
associated with a DtN operator Λ

Idea of the method

☛ Key idea Reduce the problem to a boundary value problem set in the cell Ω^i

$\Longrightarrow$ Derive suitable transparent boundary condition on Σ^i
associated with a DtN operator Λ

☛ More precisely $u^i := u_{|\Omega^i}$ solves the interior problem

$$\begin{cases} \Delta u^i + \rho u^i = f, & \text{in } \Omega^i \\ \dfrac{\partial u^i}{\partial \nu^i} + \Lambda u^i = 0 & \text{on } \Sigma^i \end{cases}$$

where Λ such that $\Lambda\phi = -\dfrac{\partial u^e(\phi)}{\partial \nu^i}$ on Σ^i with $u^e(\phi)$ the unique solution of the exterior problem

$$\begin{cases} \Delta u^e(\phi) + \rho u^e(\phi) = 0, & \text{in } \Omega^e \\ u^e(\phi) = \phi, & \text{on } \Sigma^i \end{cases}$$

Idea of the method

☛ Main steps of the method

❶ **Factorization of Λ:** involving

- a half-space DtN operator Λ^H
- a DtD operator $D_{2\pi/3}$

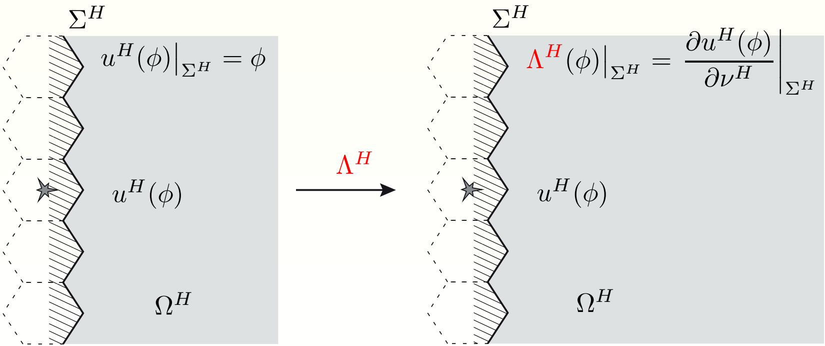
Idea of the method

☛ Main steps of the method

❶ **Factorization of Λ:** involving

- a half-space DtN operator Λ^H
- a DtD operator $D_{2\pi/3}$

❷ **Characterization of Λ^H:** Use of the periodicity property



Idea of the method

☛ Main steps of the method

❶ **Factorization of Λ:** involving

- a half-space DtN operator Λ^H
- a DtD operator $D_{2\pi/3}$

❷ **Characterization of Λ^H:** Using an adapted Floquet-Bloch transform, Λ^H can be computed via the resolution of a family of elementary cell problems and a family of Riccati operator equations.

Idea of the method

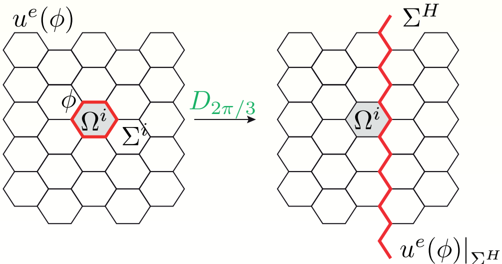
☞ Main steps of the method

❶ **Factorization of Λ:** involving

- a half-space DtN operator Λ^H
- a DtD operator $D_{2\pi/3}$

❷ **Characterization of Λ^H:** Using an adapted Floquet-Bloch transform, Λ^H can be computed via the resolution of a family of elementary cell problems and a family of Riccati operator equations.

❸ Characterization of $D_{2\pi/3}$: Use of the hexagonal symmetry property



Idea of the method

☛ Main steps of the method

❶ **Factorization of Λ:** involving

- a half-space DtN operator Λ^H
- a DtD operator $D_{2\pi/3}$

❷ **Characterization of Λ^H:** Using an adapted Floquet-Bloch transform, Λ^H can be computed via the resolution of a family of elementary cell problems and a family of Riccati operator equations.

❸ **Characterization of $D_{2\pi/3}$:** $D_{2\pi/3}$ solves an affine operator-valued equation. In practice the idea is to consider this equation using Floquet-Bloch variables $\Longrightarrow$ a set of non standard integral equations with constraints.

Restriction and extension operators

- Restriction operator $R_{2\pi/3}$

$$\begin{array}{ccc} L^2(\Sigma^i) & \to & L^2(\Sigma^0) \\ \phi & \mapsto & \phi|_{\Sigma^0} \end{array}$$

- Extension operator $E_{2\pi/3}$: Inverse of $R_{2\pi/3}$

$$\forall \phi \in L^2(\Sigma^0), \quad \left| \begin{array}{l} E_{2\pi/3}\phi\big|_{\Sigma^0} = \phi \\ E_{2\pi/3}\phi\big|_{\Theta_{2\pi/3}\Sigma^0} = \phi \circ \Theta_{-2\pi/3} \\ E_{2\pi/3}\phi\big|_{\Theta^2_{2\pi/3}\Sigma^0} = \phi \circ \Theta^2_{-2\pi/3} \end{array} \right.$$

figure

Determination of the DtN operator

Factorization of the DtN operator

Theorem

The operator Λ admits the factorization

$$\Lambda = E_{2\pi/3} \circ R^H \circ \Lambda^H \circ D_{2\pi/3}$$

where R^H is a restriction operator from Σ^H to Σ^0.

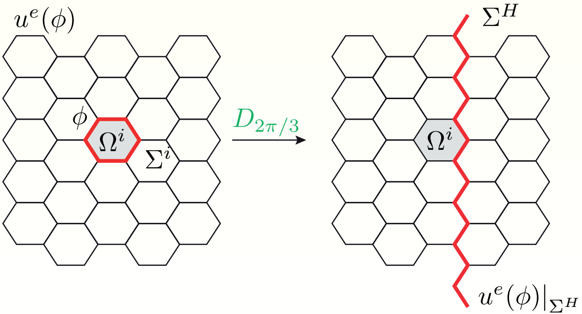
Factorization of the DtN operator

Theorem

The operator Λ admits the factorization

$$\Lambda = E_{2\pi/3} \circ R^H \circ \Lambda^H \circ D_{2\pi/3}$$

where R^H is a restriction operator from Σ^H to Σ^0.



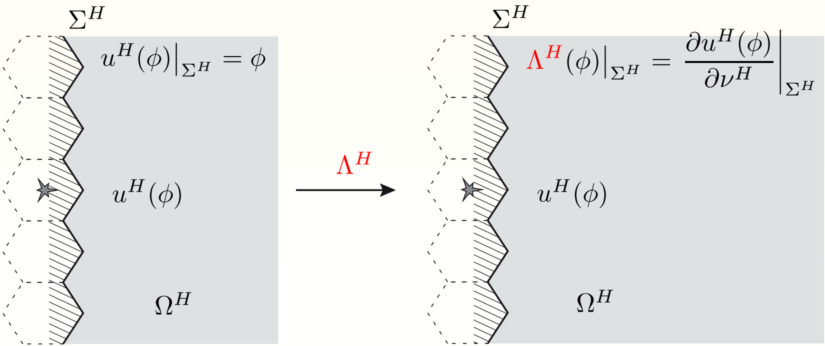
Factorization of the DtN operator

Theorem

The operator Λ admits the factorization

$$\Lambda = E_{2\pi/3} \circ R^H \circ \Lambda^H \circ D_{2\pi/3}$$

where R^H is a restriction operator from Σ^H to Σ^0.



Characterization of the half-space DtN operator

☛ Half-space problem: For any ϕ we want to compute the solution $u^H(\phi)$ of

$$(\mathcal{P}^H) \quad \begin{cases} \Delta u^H(\phi) + \rho u^H(\phi) = 0, & \text{in } \Omega^H, \\ u^H(\phi) = \phi, & \text{on } \Sigma^H, \end{cases}$$

Characterization of the half-space DtN operator

☛ Half-space problem: For any ϕ we want to compute the solution $u^H(\phi)$ of

$$(\mathcal{P}^H) \quad \begin{cases} \Delta u^H(\phi) + \rho u^H(\phi) = 0, & \text{in } \Omega^H, \\ u^H(\phi) = \phi, & \text{on } \Sigma^H, \end{cases}$$

$\Longrightarrow$ The half-space DtN operator : $\Lambda^H \phi = \left. \dfrac{\partial u^H(\phi)}{\partial \nu^H} \right|_{\Sigma^H}$.

Characterization of the half-space DtN operator

☛ Half-space problem: For any ϕ we want to compute the solution $u^H(\phi)$ of

$$(\mathcal{P}^H) \quad \begin{cases} \Delta u^H(\phi) + \rho u^H(\phi) = 0, & \text{in } \Omega^H, \\ u^H(\phi) = \phi, & \text{on } \Sigma^H, \end{cases}$$

$\Longrightarrow$ The half-space DtN operator : $\Lambda^H \phi = \left. \dfrac{\partial u^H(\phi)}{\partial \nu^H} \right|_{\Sigma^H}$.

☛ Remark: the half-space is infinite and periodic in the y-direction

$\Longrightarrow$ we use the Floquet-Bloch transform to reduce the problem to k-QP boundary data.

Characterization of the half-space DtN operator

☛ Half-space problem: For any ϕ we want to compute the solution $u^H(\phi)$ of

$$(\mathcal{P}^H) \quad \begin{cases} \Delta u^H(\phi) + \rho u^H(\phi) = 0, & \text{in } \Omega^H, \\ u^H(\phi) = \phi, & \text{on } \Sigma^H, \end{cases}$$

$$\Longrightarrow \text{The half-space DtN operator : } \Lambda^H \phi = \left.\frac{\partial u^H(\phi)}{\partial \nu^H}\right|_{\Sigma^H}.$$

☛ $k-$QP boundary data: $\phi(y+qL) = e^{\imath qkL}\phi(y)$

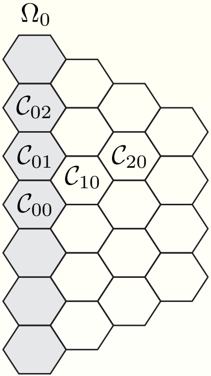
Floquet-Bloch transform $\widehat{\phi}_k(y) = \sqrt{\frac{L}{2\pi}} \sum_{m\in\mathbb{Z}} \phi(y+mL)e^{-\imath mkL}$ and its inversion formula $\phi(y) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \widehat{\phi}_k(y)dk$ imply that the solution for arbitrary boundary data is obtained by superposing the solutions for $k-$QP boundary data.

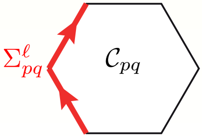
$$u^H(\phi) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} u^H\left(\widehat{\phi}_k\right)\, dk.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

Notation

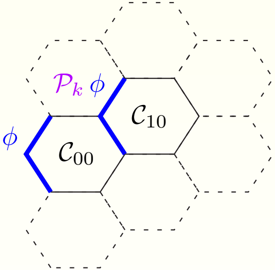
- $\mathcal{C}_{00}$: reference cell,
- $\mathbf{V}_{pq} = p\mathbf{e}_1 + q\mathbf{e}_2$,
- $\forall p \in \mathbb{N},\ q \in \mathbb{Z}, \quad \mathcal{C}_{pq} = \mathcal{C}_{00} + \mathbf{V}_{pq}$.
- $\Omega_p = \bigcup_{q\in\mathbb{Z}} \mathcal{C}_{pq}$: vertical strip containing $\mathcal{C}_{p0}$.
- Σ^{ℓ}_{pq}: oriented boundaries for a cell $\mathcal{C}_{pq}$





Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ The propagation operator



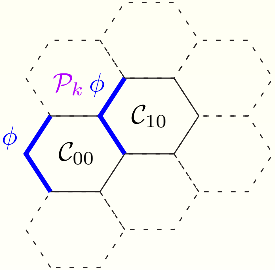
Definition

For any ϕ defined on Σ_{00}^{ℓ}, the propagation operator $\mathcal{P}_k$ is given by

$$\mathcal{P}_k\,\phi = u^H(\phi)\big|_{\Sigma_{10}^{\ell}}\,.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ The propagation operator



Definition

For any ϕ defined on Σ_{00}^{ℓ}, the propagation operator $\mathcal{P}_k$ is given by

$$\mathcal{P}_k\,\phi = u^H(E_k^{QP}\,\phi)\Big|_{\Sigma_{10}^{\ell}}\,.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☞ The propagation operator

Definition

For any ϕ defined on Σ_{00}^{ℓ}, the propagation operator $\mathcal{P}_k$ is given by

$$\mathcal{P}_k\,\phi = u^H(\,\phi)\big|_{\Sigma_{10}^{\ell}}\,.$$

Theorem

For any $k-$QP boundary data ϕ defined on Σ^H, the solution $u^H(\phi)$ of the half-space problem is given by

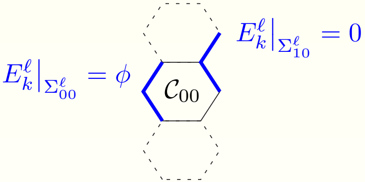
$$\forall p \in \mathbb{N},\ \ q \in \mathbb{Z}, \quad u^H(\phi)\big|_{\mathcal{C}_{pq}} = e^{\imath qkL} u^H((\mathcal{P}_k)^p\,\phi)\big|_{\mathcal{C}_{00}}\,.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Elementary problems

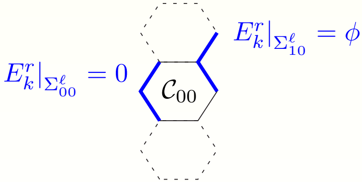
❶ $E_k^\ell(\phi)$ is the unique $k-$QP solution of

$$\begin{cases} \Delta E_k^\ell(\phi) + \rho E_k^\ell(\phi) = 0, & \text{in } \Omega_0, \\ E_k^\ell(\phi) = \phi, & \text{on } \Sigma_{00}^\ell, \\ E_k^\ell(\phi) = 0, & \text{on } \Sigma_{10}^\ell, \end{cases}$$



❷ $E_k^r(\phi_k)$ is the unique $k-$QP solution of

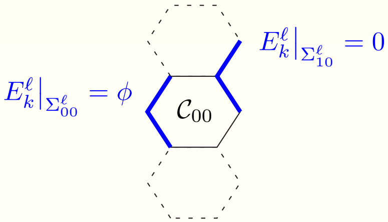
$$\begin{cases} \Delta E_k^r(\phi) + \rho E_k^r(\phi) = 0, & \text{in } \Omega_0, \\ E_k^r(\phi) = 0, & \text{on } \Sigma_{00}^\ell, \\ E_k^r(\phi) = \phi, & \text{on } \Sigma_{10}^\ell. \end{cases}$$

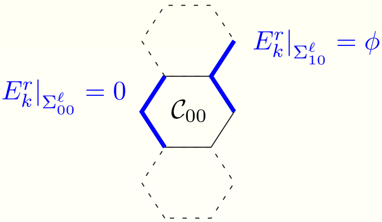


$$e_k^\ell(\phi) = E_k^\ell(\phi)\big|_{\mathcal{C}_{00}} \qquad \text{and} \qquad e_k^r(\phi) = E_k^r(\phi)\big|_{\mathcal{C}_{00}}.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Elementary problems





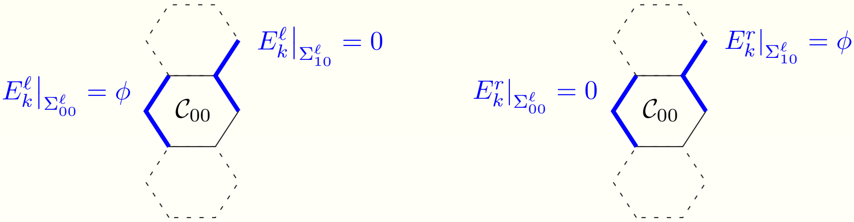
$$e_k^\ell(\phi) = E_k^\ell(\phi)\big|_{\mathcal{C}_{00}} \qquad \text{and} \qquad e_k^r(\phi) = E_k^r(\phi)|_{\mathcal{C}_{00}}\,.$$

$\Longrightarrow$ By linearity $u^H := u^H(\phi)$ satisfies

$$\begin{cases} u^H\big|_{\mathcal{C}_{00}} = e_k^\ell(\phi) + e_k^r(\mathcal{P}_k\phi), \\ u^H\big|_{\Omega_0} = E_k^\ell(\phi) + E_k^r(\mathcal{P}_k\phi). \end{cases}$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation



By linearity and periodicity:

$$u^H\big|_{\mathcal{C}_{00}} = E_k^\ell(\phi) + E_k^r(\mathcal{P}_k\phi), \qquad u^H\big|_{\mathcal{C}_{10}} = E_k^\ell(\mathcal{P}_k\phi) + E_k^r(\mathcal{P}_k^2\phi)$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation

By linearity and periodicity:

$$u^H\big|_{\mathcal{C}_{00}} = E_k^\ell(\phi) + E_k^r(\mathcal{P}_k\phi), \qquad u^H\big|_{\mathcal{C}_{10}} = E_k^\ell(\mathcal{P}_k\phi) + E_k^r(\mathcal{P}_k^2\phi)$$

The continuity of the normal derivative across Σ_{10}^ℓ yields

$$\left.\frac{\partial\left(u^H\big|_{\mathcal{C}_{00}}\right)}{\partial\nu}\right|_{\Sigma_{10}^\ell} = \left.\frac{\partial\left(u^H\big|_{\mathcal{C}_{10}}\right)}{\partial\nu}\right|_{\Sigma_{10}^\ell}$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation

By linearity and periodicity:

$$u^H\big|_{\mathcal{C}_{00}} = E_k^\ell(\phi) + E_k^r(\mathcal{P}_k\phi), \qquad u^H\big|_{\mathcal{C}_{10}} = E_k^\ell(\mathcal{P}_k\phi) + E_k^r(\mathcal{P}_k^2\phi)$$

The continuity of the normal derivative across Σ_{10}^ℓ yields

$$\left.\frac{\partial\left(u^H\big|_{\mathcal{C}_{00}}\right)}{\partial\nu}\right|_{\Sigma_{10}^\ell} = \left.\frac{\partial\left(u^H\big|_{\mathcal{C}_{10}}\right)}{\partial\nu}\right|_{\Sigma_{10}^\ell}$$

that is

$$\left(\nabla E_k^\ell(\phi) + \nabla E_k^r(\mathcal{P}_k\phi)\right)\cdot\nu|_{\Sigma_{10}^\ell} = -\left(\nabla E_k^\ell(\mathcal{P}_k\phi) + \nabla E_k^r\left(\mathcal{P}_k^2\phi\right)\right)\cdot\nu|_{\Sigma_{00}^\ell}$$

☛ Determination of $\mathcal{P}_k$: Riccati equation

$$\left(\nabla E_k^\ell(\phi) + \nabla E_k^r(\mathcal{P}_k\phi)\right)\cdot\nu|_{\Sigma_{10}^\ell} = -\left(\nabla E_k^\ell(\mathcal{P}_k\phi) + \nabla E_k^r\left(\mathcal{P}_k^2\phi\right)\right)\cdot\nu|_{\Sigma_{00}^\ell}$$

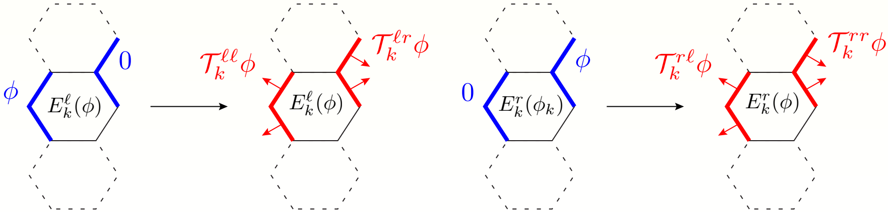
Definition

For all function ϕ on Σ_{00}^ℓ we introduce the following local DtN operators

$$\mathcal{T}_k^{\ell\ell}\phi = \nabla E_k^\ell(\phi)\cdot\nu\,\Big|_{\Sigma_{00}^\ell} \qquad \mathcal{T}_k^{\ell r}\phi_k = \nabla E_k^\ell(\phi)\cdot\nu\,\Big|_{\Sigma_{10}^\ell}$$

$$\mathcal{T}_k^{r\ell}\phi = \nabla E_k^r(\phi)\cdot\nu\,\Big|_{\Sigma_{00}^\ell} \qquad \mathcal{T}_k^{rr}\phi = \nabla E_k^r(\phi)\cdot\nu\,\Big|_{\Sigma_{10}^\ell}$$

where ν is the outgoing unit normal to $\mathcal{C}_{00}$.



☛ Determination of $\mathcal{P}_k$: Riccati equation

$$\left(\nabla E_k^{\ell}(\phi) + \nabla E_k^{r}(\mathcal{P}_k\phi)\right) \cdot \nu|_{\Sigma_{10}^{\ell}} = -\left(\nabla E_k^{\ell}(\mathcal{P}_k\phi) + \nabla E_k^{r}\left(\mathcal{P}_k^2\phi\right)\right) \cdot \nu|_{\Sigma_{00}^{\ell}}$$

Definition

For all function ϕ on Σ_{00}^{ℓ} we introduce the following local DtN operators

$$\mathcal{T}_k^{\ell\ell}\phi = \nabla E_k^{\ell}(\phi) \cdot \nu \Big|_{\Sigma_{00}^{\ell}} \qquad \mathcal{T}_k^{\ell r}\phi_k = \nabla E_k^{\ell}(\phi) \cdot \nu \Big|_{\Sigma_{10}^{\ell}}$$

$$\mathcal{T}_k^{r\ell}\phi = \nabla E_k^{r}(\phi) \cdot \nu \Big|_{\Sigma_{00}^{\ell}} \qquad \mathcal{T}_k^{rr}\phi = \nabla E_k^{r}(\phi) \cdot \nu \Big|_{\Sigma_{10}^{\ell}}$$

where ν is the outgoing unit normal to $\mathcal{C}_{00}$.

$$\Longrightarrow \quad \mathcal{T}_k^{\ell r}\phi + \mathcal{T}_k^{rr}\mathcal{P}_k\phi = -\mathcal{T}_k^{\ell\ell}\mathcal{P}_k\phi - \mathcal{T}_k^{r\ell}\mathcal{P}_k^2\phi$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation

$$\Longrightarrow \quad \mathcal{T}_k^{\ell r}\phi + \mathcal{T}_k^{rr}\mathcal{P}_k\phi = -\mathcal{T}_k^{\ell\ell}\mathcal{P}_k\phi - \mathcal{T}_k^{r\ell}\mathcal{P}_k^2\phi$$

Since ϕ is arbitrary, we get:

Theorem

The propagation operator $\mathcal{P}_k$ is the unique compact operator with spectral radius strictly less than 1 solution of the stationary Riccati equation

$$\mathcal{T}_k^{r\ell}\mathcal{P}_k^2 + (\mathcal{T}_k^{\ell\ell} + \mathcal{T}_k^{rr})\mathcal{P}_k + \mathcal{T}_k^{\ell r} = 0.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation

Theorem

The propagation operator $\mathcal{P}_k$ is the unique compact operator with spectral radius strictly less than 1 solution of the stationary Riccati equation

$$\mathcal{T}_k^{r\ell}\mathcal{P}_k^2 + (\mathcal{T}_k^{\ell\ell} + \mathcal{T}_k^{rr})\mathcal{P}_k + \mathcal{T}_k^{\ell r} = 0.$$

Proposition

For any $k-$QP boundary data ϕ, the DtN operator, associated to the half-space problem, is given by

$$\Lambda^H\phi = \mathcal{T}_k^{\ell\ell}\,\phi + \mathcal{T}_k^{r\ell}\mathcal{P}_k\,\phi.$$

Solution of $(\mathcal{P}^H)$ for quasiperiodic boundary data

☛ Determination of $\mathcal{P}_k$: Riccati equation

Theorem

The propagation operator $\mathcal{P}_k$ is the unique compact operator with spectral radius strictly less than 1 solution of the stationary Riccati equation

$$\mathcal{T}_k^{r\ell}\mathcal{P}_k^2 + (\mathcal{T}_k^{\ell\ell} + \mathcal{T}_k^{rr})\mathcal{P}_k + \mathcal{T}_k^{\ell r} = 0.$$

Proposition

For any $k-$QP boundary data ϕ, the DtN operator, associated to the half-space problem, is given by

$$\Lambda^H \phi = \mathcal{T}_k^{\ell\ell}\,\phi + \mathcal{T}_k^{r\ell}\mathcal{P}_k\,\phi.$$

For arbitrary boundary data :

$$\Lambda^H \phi = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \Lambda^H \widehat{\phi}_k dk.$$

Done

☛ We proposed a strategy to determine the DtN operator for infinite, lossy and locally perturbed hexagonal periodic media.

To be done

☛ Numerical implementation and experiments

Thank you for your attention !