Spectral and Condition Number Estimates of the Acoustic Single-Layer Operator for Low-Frequency Multiple Scattering. Part II: Dense Media*

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Abstract

The aim of this paper is to derive spectral and condition number estimates of the single-layer operator for low-frequency multiple scattering problems. This work extends the analysis initiated in [6] to dense media. Estimates are obtained first in the case of circular cylinders by Fourier analysis and are next formally adapted to disks, ellipses and rectangles in the framework of boundary element methods. Numerical simulations validating the approach are also given.

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1 Introduction

Integral equation techniques [12, 27] are an attractive and widely used tool to numerically solve acoustic, electromagnetic and elastic scattering problems. In particular, they can be used for multiple scattering configurations [25, 28] which have many applications in physics and mechanics [8, 14, 15, 16, 17, 20, 21, 24, 26]. A classical way to solve an integral equation is to use an iterative Krylov subspace solver (like e.g. the GMRES [29]) coupled to a Matrix-Vector acceleration scheme like the Fast Multipole Method [13, 18, 28]. A well-known property is that the convergence rate of the iterative solver is related to the condition number and spectral distribution of the integral equation. For this reason, understanding the spectral properties helps in building suitable preconditioners for integral equations. Spectral estimates have already been obtained for single scattering configurations. We refer to [4, 5, 6, 22, 23] for complete studies involving circular cylinders, convex and non convex structures [7, 10] as well as open surfaces or waveguides [2, 3, 11]. Multiple scattering problems are more complex in the sense that interactions between obstacles must be considered in the analysis. In a first part [6], we derived low-frequency spectral and condition number estimates of the single-layer potential for many distant obstacles (disks, ellipses and rectangles). The aim of this second part is to complete the previous results when the scatterers are close (dense media).

The plan of the paper is the following. In Section 2, we briefly introduce the single-layer operator and its spectral formulation. We also recall the analytical formula of the single-layer operator when the obstacle is a collection of circular cylinders. Section 3 explains how to obtain eigenvalues and condition number estimates of the single-layer operator for disks. Section 4 provides some extensions of the results to circular, elliptical and rectangular cylinders in the framework of linear boundary element methods. Finally, Section 5 gives a conclusion and some perspectives.

2 The single-layer operator for multiple scattering

2.1 Single-layer potential for multiple scattering

Let $\Omega^-$ be a (possibly multi-connected) bounded open set of the two-dimensional space. Its boundary is denoted by $\Gamma$ and $\mathbf{n}$ is the outwardly directed unit normal to the scatterer. The propagation domain $\Omega^+ = \mathbb{R}^2 \setminus \overline{\Omega^-}$ is supposed to be connected. Since we want to analyze the spectrum of the single-layer potential, we consider the following spectral problem: find the pair $(\rho, \mu) \in L^2(\Gamma) \times \mathbb{C}$ solution to

$$L \rho = \mu \rho, \quad \text{in } H^{1/2}(\Gamma),$$

where the single-layer boundary integral operator $L$ is given by

$$L : \quad H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$$

$$\rho \quad \rightarrow \quad L \rho, \quad \forall x \in \Gamma, \quad L \rho (x) = \int_{\Gamma} G(x, y) \rho(y) d\Gamma(y),$$

with the Green’s function $G(\cdot, \cdot)$

$$\forall x, y \in \mathbb{R}^2, x \neq y, \quad G(x, y) = \frac{i}{4} H_0^{(1)}(k \|x - y\|),$$

setting $H_0^{(1)}$ as the zeroth order Hankel’s function of the first kind and $\|x\| = (x^2_1 + x^2_2)^{1/2}$. We refer to [27] concerning the functional spaces. For computational purposes, we introduce the variational

...
formulation of (1) on $L^2(\Gamma)$

$$\begin{align*}
\text{Find } (\rho, \mu) \in L^2(\Gamma) \times \mathbb{C} \text{ such that }
\forall \Phi \in L^2(\Gamma), \quad (L\rho, \Phi)_{L^2(\Gamma)} = \mu (\rho, \Phi)_{L^2(\Gamma)},
\end{align*}$$

(2)

where the inner product $(\cdot, \cdot)_{L^2(\Gamma)}$ is defined by

$$\forall (f, g) \in L^2(\Gamma) \times L^2(\Gamma), \quad (f, g)_{L^2(\Gamma)} = \int_{\Gamma} f \bar{g} \, d\Gamma,$$

and $\bar{g}$ is the complex conjugate of $g$. The multiple scattering problem consists in an obstacle $\Omega^-$ with $M$ components $\Omega^-_1, \ldots, \Omega^-_M$ with respective boundaries $\Gamma_1, \ldots, \Gamma_M$. Then, we have $L^2(\Gamma) = L^2(\Gamma_1) \times \ldots \times L^2(\Gamma_M)$ and

$$(f, g)_{L^2(\Gamma)} = \sum_{p=1}^{M} \int_{\Gamma_p} f_p \bar{g}_p \, d\Gamma_p,$$

where $f = (f_1, \ldots, f_M)$ and $g = (g_1, \ldots, g_M)$ are functions of the $L^2(\Gamma)$ space.

### 2.2 Expression of the single-layer potential for $M$ circular cylinders

We assume now that $\Omega^-$ is the union of $M$ strictly disjoint (no sticky case) circular cylinders $\Omega^-_p$, $p = 1, \ldots, M$. The boundary $\Gamma_p$ of $\Omega^-_p$ is a circle with center $O_p$ and radius $a_p$, the boundary of $\Omega^-$ being denoted by $\Gamma$. The explicit expression of the single-layer potential in the case of several disks has been obtained e.g. in [6]. Let us recall that, for $p = 1, \ldots, M$, any point $x$ of $\mathbb{R}^2$ is given by its polar coordinates linked to the obstacle $\Omega^-_p$

$$r_p(x) = O_p x, \quad r_p(x) = \|r_p(x)\|, \quad \theta_p(x) = \text{Angle}(\overrightarrow{O_1 x}, O_p x).$$

If $O$ is the origin and if $O_p$ is the center of the disc $\Omega^-_p$, we set

$$b_p = OO_p, \quad b_p = \|b_p\|, \quad \alpha_p = \text{Angle}(\overrightarrow{O_1 b_p}),$$

and, for $q = 1, \ldots, M$, $p \neq q$,

$$b_{pq} = O_q O_p, \quad b_{pq} = \|b_{pq}\|, \quad \alpha_{pq} = \text{Angle}(\overrightarrow{O_1 b_{pq}}).$$

By introducing $\mathcal{B} = (\Phi_p^q)_{p=1,\ldots,M,m\in\mathbb{Z}}$ as an orthonormal basis of $L^2(\Gamma)$ [6], we can decompose $\rho$ as: $\rho = \sum_{q=1}^{M} \sum_{n\in\mathbb{Z}} \rho^q_n \Phi_n^q$. The spectral problem then writes

$$\mathcal{L} \rho = \mu \rho,$$

with

$$
\begin{pmatrix}
\mathcal{L}^0_{1,1} & \mathcal{L}^0_{1,2} & \cdots & \mathcal{L}^0_{1,M} \\
\mathcal{L}^1_{2,1} & \mathcal{L}^1_{2,2} & \cdots & \mathcal{L}^1_{2,M} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{L}^M_{M,1} & \mathcal{L}^M_{M,2} & \cdots & \mathcal{L}^M_{M,M}
\end{pmatrix}
\begin{pmatrix}
\rho^0 \\
\rho^1 \\
\vdots \\
\rho^M
\end{pmatrix}
= \mu
\begin{pmatrix}
\rho^0 \\
\rho^1 \\
\vdots \\
\rho^M
\end{pmatrix},
$$

(3)
where each infinite block $\tilde{L}^{p,q}$, $1 \leq p, q \leq M$ of the matrix $\tilde{L}$ has coefficients

$$\tilde{L}^{p,q}_{m,n} = (L \Phi^p_m, \Phi^q_n)_{L^2(\Gamma)}, \quad \forall m, n \in \mathbb{Z}. \tag{4}$$

The coefficients (4) of the single-layer potential, for two objects $p$ and $q$, with $p, q = 1, \ldots, M$, for two Fourier modes $m, n \in \mathbb{Z}$, are given by

$$\tilde{L}^{p,q}_{m,n} = \begin{cases} \frac{i\pi a_p}{2} J_m(ka_p)H^{(1)}_m(ka_p)\delta_{mn} & \text{if } p = q, \\ \frac{i\pi \sqrt{a_p a_q}}{2} J_m(ka_p)S_{nm}(b_{pq})J_n(ka_q) & \text{otherwise.} \end{cases} \tag{5}$$

Symbol $\delta_{mn}$ denotes the Kronecker’s delta function, equal to 1 if $m = n$ and 0 otherwise. The quantity $S_{nm}(b_{pq})$ is given by: $S_{nm}(b_{pq}) = H^{(1)}_{n-m}(kb_{pq})e^{i(n-m)\alpha_{pq}}$, for $p, q = 1, \ldots, M, p \neq q$ and $m, n \in \mathbb{Z}$. In practice, a finite dimensional projection is required. This is done by truncating system (3) by keeping, for each Fourier series $(\Phi^p_m)_{m \in \mathbb{Z}}$, $p = 1, \ldots, M$, $2N_p + 1$ modes such that:

$-N_p \leq m \leq N_p$. The resulting truncated system is denoted by $L \rho = -U$, with $L$:

$$L = \begin{pmatrix} L^{1,1} & L^{1,2} & \cdots & L^{1,M} \\ L^{2,1} & L^{2,2} & \cdots & L^{2,1} \\ \vdots & \vdots & \ddots & \vdots \\ L^{M,1} & L^{M,2} & \cdots & L^{M,M} \end{pmatrix} \begin{pmatrix} \rho^1 \\ \rho^2 \\ \vdots \\ \rho^M \end{pmatrix} = \mu \begin{pmatrix} \rho^1 \\ \rho^2 \\ \vdots \\ \rho^M \end{pmatrix},$$

where each block $L^{p,q}$, of size $(2N_p + 1) \times (2N_q + 1)$, for $p, q = 1, \ldots, M$, of matrix $L$ has coefficients (5): $L^{p,q}_{m,n} = \tilde{L}^{p,q}_{m,n}$, for $-N_p \leq m \leq N_p$ and $-N_q \leq q \leq N_q$.

To simplify, we denote by $I$ the set of all the couple of indexes $(p, m)$ (or $(q, n)$):

$$I = \{(p, m) \in \mathbb{Z} \text{ such that } 1 \leq p \leq M \text{ and } -N_p \leq m \leq N_p\}.$$

## 3 Low frequency condition number estimates of the single-layer potential for close obstacles: the case of circular cylinders

In this Section, we assume that we have a dense media: the obstacles are small and close. In other words, setting $b = \min_{p,q=1,\ldots,M, p \neq q} b_{pq}$, we are interested in the asymptotic regime where both $ka$ and $kb$ tend towards 0 at the same speed. To analyze this regime, we potentially have two methods. The first one consists in choosing a fixed wave number $k$ and applying a dilation to the geometrical configuration. The second one, that will be followed here, is to fix the geometry and to let $k$ tends towards 0. As a result, the radii $a_p$, the centers of the obstacles $O_p$ as well as the distances between the centers $b_{pq}$ are supposed to be constant.

Like in [6], we consider the limit matrix approach to derive some estimates of the eigenvalues $\mu_{\min}$ and $\mu_{\max}$, respectively, with smallest and largest modulus, of the matrix $L$ when $k$ tends towards zero. In particular, we show that $\mu_{\min}$ can still be approximated by the eigenvalue with smallest modulus related to single scattering. As a by-product, these approximations allow us to derive condition number estimates.
3.1 The limit matrix approach

Our approach follows the one introduced in the first part [6] for distant obstacles (Section 5). Indeed, we develop an asymptotic analysis of the coefficients of $L$ when $k$ tends towards 0. Then, by using these approximations, we build a limit matrix $L_0$ and we get the following relation involving $L$ and $L_0$: $L = L_0 + O\left(k^2 \ln(k)\right)$. We finally approximate the spectrum of $L$ by the one of $L_0$.

Let us begin by analyzing each diagonal block $L_{p,p}$, for $p = 1, \ldots, M$. Let us recall that these matrices are diagonal and that their coefficients $L_{p,p,m,m}$ write, for $m = -N_p, \ldots, N_p$,

$$L_{p,p,m,m} = \frac{i\pi a_p}{2} J_m(ka_p) H_m^{(1)}(ka_p).$$

Lemma 1 in [6] gives the asymptotics of these coefficients when $ka_p \to 0$ and it remains true when only the wave number $k$ tends to 0. We summarize these results in the following Lemma.

**Lemma 1.** For all $(p,m) \in I$ and when $k \to 0$, the following result holds

$$L_{p,p,m,m} = \begin{cases} (L_0)_{p,p,0,0} + O\left((k)^2 \ln(k)\right) & \text{for } m = 0, \\ (L_0)_{p,p,m,m} + O\left((k)^2\right) & \text{for } m \neq 0, \end{cases}$$

setting

$$(L_0)_{p,p,m,m} = \begin{cases} -a_p \left[\ln\left(\frac{ka_p}{2}\right) + \gamma\right] + \frac{\pi a_p}{2} & \text{for } m = 0, \\ \frac{a_p}{2|m|} & \text{for } m \neq 0, \end{cases}$$

where $\gamma = 0.5772 \ldots$ is the Euler constant.

Following the approach in [6], for any $p = 1, \ldots, M$, we build the diagonal matrix $(L_0)^{p,p}$, of size $(2N_p + 1) \times (2N_p + 1)$, defined by

$$(L_0)^{p,p} = \begin{pmatrix} (L_0)^{p,p}_{-N_p,-N_p} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & (L_0)^{p,p}_{N_p,N_p} \end{pmatrix},$$

where the coefficients $(L_0)^{p,p}_{m,m}$ are given by (7). Relation (6) between the coefficients of $L^{p,p}$ and the ones of $(L_0)^{p,p}$ imply that, when $k \to 0$, we have $L^{p,p} = (L_0)^{p,p} + O\left(k^2 \ln(k)\right)$, for $p = 1, \ldots, M$. Let us remark that the limit matrix $(L_0)^{p,p}$ is the same as for distant obstacles. This is expected since the main difference lies in the parameters $kb_{pq}$ (the distance between $\Omega_p^-$ and $\Omega_q^-$). These terms do not appear in the diagonal blocks $L^{p,p}$ but only in the off-diagonal blocks $L^{p,q}$, for $p \neq q$.

The asymptotic behaviors of the off-diagonal blocks coefficients $L^{p,q}_{m,n}$ differ according to the indices $m$ and $n$. We split each block $L^{p,q}_{m,n}$ into 5 zones, labeled from 0 to 4, as reported on Figure 1. We next develop an asymptotic analysis of the coefficients $L^{p,q}_{m,n}$ when $k \to 0$ with respect to each zone. The results are summarized in the following Lemma.
Figure 1: Decomposition of $L_{p,q}^{m,n}$ into five different zones, for $p \neq q$.

**Lemma 2.** Let $(p,m) \in I$ and $(q,n) \in I$ with $p \neq q$. When $k$ tends towards 0, the coefficients $L_{m,n}^{p,q}$ of the matrix $L$ have the following asymptotic behavior

- **Zone 0**: $(m = n = 0)$
  $$L_{0,0}^{p,q} = -\sqrt{a_p a_q} \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + \frac{\pi \sqrt{a_p a_q}}{2} + O \left( k^2 \ln(k) \right).$$

- **Zone 1**: $(mn \leq 0$ and $(m,n) \neq (0,0))$
  $$L_{m,n}^{p,q} = \begin{cases} 
  i(-1)^m \frac{\sqrt{a_p a_q}}{2} \frac{1}{|m|!|n|!} \left( \frac{a_p}{b_{pq}} \right)^{|m|} \left( \frac{a_q}{b_{pq}} \right)^{|n|} e^{i(n-m)\alpha_{pq}} + O \left( k^2 \ln(k) \right) & \text{if } |m|+|n|=1, \\
  i(-1)^m \frac{\sqrt{a_p a_q}}{2} \frac{1}{|m|!|n|!} \left( \frac{a_p}{b_{pq}} \right)^{|m|} \left( \frac{a_q}{b_{pq}} \right)^{|n|} e^{i(n-m)\alpha_{pq}} + O \left( k^2 \ln(k) \right) & \text{otherwise}. 
  \end{cases}$$

- **Zone 2**: $(n = m, m \neq 0$ and $n \neq 0)$
  $$L_{m,m}^{p,q} = O \left( k^2 |m| \ln(k) \right).$$

- **Zone 3**: ($(0 > m > n$ or $0 < m < n)$)
  $$L_{m,n}^{p,q} = O \left( k^2 |m| \right).$$

- **Zone 4**: ($(0 > n > m$ or $0 < n < m)$)
  $$L_{m,n}^{p,q} = O \left( k^2 |n| \right).$$

**Proof.** We prove the results zone by zone.

- **Zone 0**, $m = n = 0$. In this zone, the coefficients $L_{m,n}^{p,q}$ write down
  $$L_{0,0}^{p,q} = i \frac{\pi \sqrt{a_p a_q}}{2} J_0(k\alpha_p) H_0^{(1)}(kb_{pq}) J_0(k\alpha_q). \quad (9)$$
Let us recall that the first kind Hankel’s functions of order zero have the following expansion when $k \to 0$ (see relations (9.1.8) and (9.1.13) in [1])

$$H_0^{(1)}(kb_{pq}) = 1 + i \frac{2}{\pi} \ln \left( \frac{kb_{pq}}{2} \right) + O \left( k^2 \ln(k) \right),$$

and the Bessel’s function of order $m \in \mathbb{N}$ is such that (Eq. (9.1.10) in [1])

$$J_m(ka_p) = \frac{1}{m!} \left( \frac{ka_p}{2} \right)^m + O \left( k^{m+2} \right).$$

By injecting (11) and (10) into (9), we obtain

$$\mathbb{L}_{0,0}^{p,q} = i \frac{\pi \sqrt{a_p a_q}}{2} \left[ 1 + O \left( k^2 \right) \right] \left[ 1 + i \frac{2}{\pi} \ln \left( \frac{kb_{pq}}{2} \right) + O \left( k^2 \ln(k) \right) \right] \left[ 1 + O \left( k^2 \right) \right],$$

that is, after some developments,

$$\mathbb{L}_{0,0}^{p,q} = -i \frac{\sqrt{a_p a_q}}{2} \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + i \frac{\pi \sqrt{a_p a_q}}{2} + O \left( k^2 \ln(k) \right),$$

which is the expected relation.

- **Zone 1** ($mn \leq 0$ and $(m,n) \neq (0,0)$). Let us introduce the sign function, denoted by sgn, such that for any $n \in \mathbb{Z}$, $\text{sgn}(n) = 1$ if $n \geq 0$ and $\text{sgn}(n) = -1$ if $n < 0$. Using some properties of the special functions, we can write the coefficients $\mathbb{L}_{m,n}^{p,q}$ under the following form

$$\mathbb{L}_{m,n}^{p,q} = (\text{sgn}(m))^m (\text{sgn}(n))^n (\text{sgn}(n-m))^n - m \cdot \frac{i \pi \sqrt{a_p a_q}}{2} e^{i(n-m)\alpha_{pq}} J_{|m|}(ka_p) H_{|n-m|}^{(1)}(kb_{pq}) J_{|n|}(ka_q).$$

(12)

Let us note that the indices of the Bessel’s and Hankel’s functions are now positive. For the coefficients of zone 1, the indices $m$ and $n$ satisfy $mn \leq 0$. From the one hand, we have $|n-m| = |n| + |m|$, and, from the other hand, $\text{sgn}(m) = -\text{sgn}(n)$ and $\text{sgn}(n-m) = \text{sgn}(n)$, which imply that

$$(\text{sgn}(m))^m (\text{sgn}(n))^n (\text{sgn}(n-m))^n - m = (-\text{sgn}(n))^m (\text{sgn}(n))^n (\text{sgn}(n))^{n+m} = (-1)^m.$$

The coefficients $\mathbb{L}_{m,n}^{p,q}$ in this zone can be written from (12),

$$\mathbb{L}_{m,n}^{p,q} = (-1)^m \frac{i \pi \sqrt{a_p a_q}}{2} e^{i(n-m)\alpha_{pq}} J_{|m|}(ka_p) H_{|n|}^{(1)}(kb_{pq}) J_{|n|}(ka_q).$$

(13)

Let us recall the asymptotic expansions of the first kind Hankel’s functions of order $m > 0$ when $k \to 0$ (see relations (9.1.9) and (9.1.11) in [1])

$$H_m^{(1)}(kb_{pq}) = \frac{(m-1)!}{\pi} \left( \frac{kb_{pq}}{2} \right)^{-m} + O \left( f_m(k) \right),$$

where the functions $f_m$ are defined by: $f_1(k) = k \ln(k)$ and $f_m(k) = k^{2-m}$ for $m \geq 2$. Next, we use the Bessel’s and Hankel’s functions expansions (11) and (14) into the expression (13) of the
coefficients \( \mathbb{L}_{m,n}^{p,q} \) to get

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \sqrt{a_p a_q} e^{i(n-m)\alpha_{pq}} \left[ \frac{1}{|m|!} \left( \frac{ka_p}{2} \right)^{|m|} + O\left( k^{m+2} \right) \right]
\]

\[
\times \left[ \left( \frac{|m| + |n| - 1}{\pi} \right)^{|m|-|n|} \left( \frac{kb_p}{2} \right)^{-|m|} + O\left( f_{|m|+|n|}(k) \right) \right] \left[ \frac{1}{|n|!} \left( \frac{ka_q}{2} \right)^{|n|} + O\left( k^{n+2} \right) \right].
\]

Let us note that we can use the asymptotic relation (14) of the Hankel’s functions of order \(|m| + |n|\). Indeed, for the coefficients in this zone, the indices \((m, n)\) satisfy \(mn \leq 0\) and \((m, n) \neq (0, 0)\), which in particular imply that \(|m| + |n| \neq 0\). Hence we study the Hankel’s functions with non null index.

We then develop the previous relation to obtain

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \sqrt{a_p a_q} e^{i(n-m)\alpha_{pq}} \left( |m| + |n| - 1 \right)^{|m|-|n|} \left( \frac{ka_p}{2} \right)^{|m|} \left( \frac{ka_q}{2} \right)^{|n|} \left( \frac{2}{kb_p} \right)^{|m|+|n|}
\]

\[
+ O\left( k^{n+2} \right) + O\left( k^{m+2} \right).
\]

After some simplifications, we have

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \sqrt{a_p a_q} \left( |m| + |n| - 1 \right)^{|m|-|n|} \left( \frac{a_p}{b_p} \right)^{|m|} \left( \frac{a_q}{b_q} \right)^{|n|} e^{i(n-m)\alpha_{pq}} + O\left( k^2 \right)
\]

\[
+ O\left( k^{n+2} \right).
\]

From the definition of the functions \( f_{|m|+|n|} \), we have

\[
k^{m+|n|} f_{|m|+|n|}(k) = \begin{cases} k^2 \ln(k) & \text{if } |m| + |n| = 1, \\ k^2 & \text{otherwise.} \end{cases}
\]

By injecting these relations into (15), we obtain the expected relation.

- **Zone 2**, \((m = m, n \neq 0 \text{ and } n \neq 0)\). To prove the relations in the zones 2, 3 and 4, we only need to analyze the modulus of the coefficient \( \mathbb{L}_{m,n}^{p,q} \), that is, in the general case

\[
\mathbb{L}_{m,n}^{p,q} = \sqrt{a_p a_q} \left| J_{|m|}(ka_p) \right| \left| H_{|n-m|}^{(1)}(kb_p) \right| \left| J_{|n|}(ka_q) \right|. \tag{16}
\]

For zone 2, we have \(|n-m| = 0\), with \(m \neq 0\) and \(n \neq 0\). The modulus of the coefficient \( \mathbb{L}_{m,n}^{p,q} \) in this zone is then given by

\[
\mathbb{L}_{m,n}^{p,q} = \sqrt{a_p a_q} \left| J_{|m|}(ka_p) \right| \left| H_{0}^{(1)}(kb_p) \right| \left| J_{|n|}(ka_q) \right|. \]

We next use the asymptotics (11) and (10) of the Bessel’s and zeroth-order Hankel’s functions to obtain

\[
\mathbb{L}_{m,n}^{p,q} = \left| \frac{\pi \sqrt{a_p a_q}}{2} \left( \frac{ka_p}{2} \right)^{|m|} + O\left( k^{m+2} \right) \right| \left[ 1 + \frac{2}{\pi} \left[ \ln \left( \frac{kb_p}{2} \right) + \gamma \right] + O\left( k^2 \ln(k) \right) \right]
\]

\[
\times \left[ \left( \frac{ka_q}{2} \right)^{|n|} + O\left( k^{n+2} \right) \right].
\]
This directly provides the relation: \( L_{m,n}^{p,q} = O \left( k^{2|m|} \ln(k) \right) \).

- **Zone 3**: since \((0 > m > n)\) or \((0 < m < n)\), we have \(|n-m| = |n| - |m|\). The expression (16) of the modulus of the coefficient \( L_{m,n}^{p,q} \) is then

\[
|L_{m,n}^{p,q}| = \frac{\sqrt{a_p a_q}}{2} \left| J_{|n|}(ka_p) \right| \left| H_{|n|-|m|}^{(1)}(kb_{pq}) \right| \left| J_{|n|}(ka_q) \right| .
\]

We use the asymptotics (11) and (14) of the Bessel’s and Hankel’s functions to get

\[
|L_{m,n}^{p,q}| = \left| \frac{\pi}{2} \frac{1}{|n|!} \left( \frac{ka_p}{2} \right)^{|m|} + O \left( k^{|m|+2} \right) \right|
\]

\[
\left[ \frac{(|n| - |m| - 1)!}{\pi} \left( \frac{kb_{pq}}{2} \right)^{-|n|+|m|} + O \left( f_{|n|-|m|}(k) \right) \right] \left[ \frac{1}{|n|!} \left( \frac{ka_q}{2} \right)^{|n|} + O \left( k^{|n|+2} \right) \right] .
\]

Hence we directly obtain: \( L_{m,n}^{p,q} = O \left( k^{2|m|} \right) \).

- **Zone 4**: since \((0 > n > m)\) or \((0 < n < m)\), we have: \(|n-m| = |m| - |n|\). Let us point out that, by changing the role of \( m \) and \( n \), we recover the results of zone 3. Thus, the proof developed above can be adapted here to directly obtain \( L_{m,n}^{p,q} = O \left( k^{2|m|} \right) \). \(\square\)

For any \(1 \leq p \neq q \leq M\), let us introduce the limit matrix \((L^0)^{p,q}\) with the same size as \(L^{p,q}\), whose coefficients are the first terms of the asymptotic expansion given by Lemma 2. More precisely, the coefficients \((L^0)^{p,q}_{m,n}\) of \((L^0)^{p,q}\) satisfy, for \(-N_p \leq m \leq N_p\) and \(-N_q \leq n \leq N_q\)

\[
(L^0)^{p,q}_{m,n} = \begin{cases} 
-\sqrt{\alpha_p \alpha_q} \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + i \frac{\pi \sqrt{\alpha_p \alpha_q}}{2} & \text{if } (m,n) = (0,0), \\
(-1)^m i \frac{\alpha_p \alpha_q}{2} \left( \frac{|m| + |n| - 1)!}{|m|!|n|!} \right) \left( \frac{a_p}{b_{pq}} \right)^{|m|} \left( \frac{a_q}{b_{pq}} \right)^{|n|} e^{i(n-m)\alpha_{pq}} & \text{if } (m,n) \neq (0,0) \text{ and } mn \leq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

(17)

Moreover, from Lemma 2, we have the following relation between the block \(L^{p,q}\) and the matrix \((L^0)^{p,q}\)

\[
L^{p,q} = (L^0)^{p,q} + O \left( k^2 \ln(k) \right), \quad \text{for } p \neq q .
\]

(18)

Let us now introduce the block matrix \(L^0\) containing each submatrix \((L^0)^{p,q}\) and defined by

\[
L^0 = \begin{pmatrix}
(L^0)^{1,1} & (L^0)^{1,2} & \cdots & (L^0)^{1,M} \\
(L^0)^{2,1} & (L^0)^{2,2} & \cdots & (L^0)^{2,M} \\
\vdots & \vdots & \ddots & \vdots \\
(L^0)^{M,1} & (L^0)^{M,2} & \cdots & (L^0)^{M,M}
\end{pmatrix}.
\]

By using (8) and (18), the following proposition holds.

**Proposition 1.** When \(k\) tends towards zero, the truncated matrix \(L\) of the single-layer operator \(L\) satisfies the relation

\[
L = L^0 + O \left( k^2 \ln(k) \right).
\]

(19)
Figure 2: Non null coefficients of the matrix $L^0$ for a two obstacles configuration. The grey zones correspond to the non zero coefficients.

To visualize the relatively sparse structure of $L^0$, we represent on Figure 2 in grey color the non zero coefficients (skeleton of $L^0$) in the case of two circular scatterers.

From now, let us respectively denote by $(\mu_{p,m})_{(p,m)\in I}$ and $(\mu^0_{p,m})_{(p,m)\in I}$ the eigenvalues of $L$ and $L^0$. For $(p,m)\in I$, we moreover assume that

$$\mu_{p,m} \simeq (\mu^0_{p,m}),$$

which is coherent with (19). To motivate our approach, we compare the eigenvalues of $L$ and $L^0$ on Figures 3(b)-3(d), for 30 randomly distributed disks of radius 0.1 in $[0,4]^2$ (see the example of Figure 3(a)) and $k = 0.1$. For $p, q = 1, \ldots, M$, with $p \neq q$, $N_p = 5$ and the intercenter distance $b_{pq}$ between $\Omega_p$ and $\Omega_q$ is within the range 0.33 and 4.59. The eigenvalues have been numerically computed by the Matlab function `eig`. We observe that the approximation of the eigenvalues $(\mu^p_{m})_{(p,m)\in I}$ of $L$ by the eigenvalues $(\mu^0_{m})_{(p,m)\in I}$ of $L^0$ is satisfactory. Let us denote by $\mu_{\text{max}}$ and $\mu^0_{\text{max}}$ the eigenvalues with largest modulus of respectively $L$ and $L^0$. On Figure 3(b), we observe that they are very close. This result is numerically confirmed since one gets: $\mu_{\text{max}} \simeq 5.393 + 4.623i$ and $\mu^0_{\text{max}} \simeq 5.494 + 4.695i$, which means that: $|\mu_{\text{max}}| \simeq 7.103$ and $|\mu^0_{\text{max}}| \simeq 7.227$. Furthermore, the eigenvalues with smallest modulus of $L$ and $L^0$, respectively denoted by $\mu_{\text{min}}$ and $\mu^0_{\text{min}}$, are also very close as seen on Figure 3(d). The numerical values confirm this result: $\mu_{\text{min}} \simeq 0.01 + 10^{-13}i$, and $\mu^0_{\text{min}} \simeq 0.01 + 10^{-13}i$.

This example confirms that our approach seems reasonable. Further simulations have been performed and show that the eigenvalues of $L^0$ are close to the ones associated with $L$. To simplify the computations, we assume that, for any $p = 1, \ldots, M$, we have: $a_p = a$ and $N_p = N$. Let us recall that, for $p = 1, \ldots, M$, the diagonal blocks $(L^0)^{p,p}$ of $L^0$ do not depend on $p$. Moreover, we
Figure 3: Comparison of the eigenvalues of $L$ and $L^0$ for 30 randomly distributed disks of radius $a = 0.1$ in $[0, 4]^2$, $N_p = 5$ for each obstacle ($k = 0.1$ and $0.33 \leq b_{pq} \leq 4.59$).

have, for $-N \leq m \leq N$ and $p, q = 1, \ldots, M$

$$
(\mathbb{L}^0)_{p,p}^{m,m} = (\mathbb{L}^0)_{p,p}^{m,-m} = (\mathbb{L}^0)_{q,q}^{m,m} = (\mathbb{L}^0)_{q,q}^{m,-m}.
$$

These terms are now denoted by $\hat{\mathbb{L}}_m$

$$
\forall m = 0, \ldots, N, \quad \hat{\mathbb{L}}_m = (\mathbb{L}^0)_{p,p}^{m,m} = \begin{cases} 
-a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{i \pi a}{2}, & \text{if } m = 0, \\
\frac{a}{2m}, & \text{otherwise}.
\end{cases}
$$

### 3.2 Estimates of the eigenvalue with smallest modulus

Let us now estimate the eigenvalue $\mu_{\min}^0$, with smallest modulus of $\mathbb{L}^0$. Like for the case of distant obstacles [6], an approach based on the Gershgorin-Hadamard discs (Theorems 1 and 2 in [6]) has been developed but does not provide any estimate of the eigenvalue with smallest modulus. In
addition, the obtained approximation for the largest eigenvalue is inaccurate, most particularly for many obstacles.

Like for distant scatterers [6], we show that the eigenvalue of $L_0$ (and next $L_0$) with smallest modulus can be approximated correctly by the associated single scattering eigenvalue, that is: $\hat{L}_N = \frac{a}{2N}$, with multiplicity $2M$. To prove this result, we build an approximate eigenvector of $L_0$ for the approximate eigenvalue $L_{\lvert m \rvert}$, for large $\lvert m \rvert$. Let us introduce the vector $X_{(p,m)}$, for $(p,m) \in I$, given by

$$X_{(p,m)} = \begin{pmatrix} X_{1_{(p,m)}} \\ X_{2_{(p,m)}} \\ \vdots \\ X_{M_{(p,m)}} \end{pmatrix}.$$  

Each block $X_{q_{(p,m)}} = (X_{q_{(p,m)}})_{n=-N,...,N}$, for $q = 1, \ldots, M$, has size $2N + 1$ and its components are defined by the following relations

$$X_{p_{,n}}^{(p,m)} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } q = p, \quad (21)$$

and

$$X_{q_{,n}}^{(p,m)} = 0, \quad \text{for } q \neq p. \quad (22)$$

Hence, the blocks $X_{q_{(p,m)}}$ are zero for $q \neq p$, and the block $X_{p_{(p,m)}} = (X_{p_{(p,m)}})_{n=-N,...,N}$ has only one nonzero component localized at $n = m$ (and equal to 1). Let us remark that $X_{(p,m)}$ is also normalized for both the infinity and euclidian norms. We introduce now $Y_{(p,m)}$ as the vector resulting from the matrix-vector product between $L_0$ and $X_{(p,m)}$

$$Y_{(p,m)} = L_0 X_{(p,m)} = \begin{pmatrix} Y_{1_{(p,m)}} \\ Y_{2_{(p,m)}} \\ \vdots \\ Y_{M_{(p,m)}} \end{pmatrix} = \begin{pmatrix} (L_0)^{1,p} X_{p_{(p,m)}} \\ (L_0)^{2,p} X_{p_{(p,m)}} \\ \vdots \\ (L_0)^{M,p} X_{p_{(p,m)}} \end{pmatrix}.$$  

Let us focus on the vectorial block $Y_{p_{(p,m)}}$. From the definition (8) of the extracted matrix $(L_0)^{p,p}$, the coefficients of the vector $Y_{p_{(p,m)}} = (Y_{p_{,n}}^{(p,m)})_{n=-N,...,N}$ satisfy

$$Y_{p_{,n}}^{(p,m)} = \begin{cases} (L_0)^{p,p}_{m,m} = \hat{L}_{\lvert m \rvert} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } n \in \mathbb{Z}$$

This means that, from relation (21), we have

$$Y_{p_{(p,m)}} = \hat{L}_{\lvert m \rvert} X_{p_{(p,m)}}.$$  

Let us now analyze the vectors $Y_{q_{(p,m)}} = (L_0)^{q,p} X_{p_{(p,m)}}$, for $q = 1, \ldots, M$ and $q \neq p$. From the particular structure of $X_{p_{(p,m)}}$ (see (21)), the coefficients $Y_{q_{,n}}^{(p,m)}$ of vector $Y_{q_{(p,m)}}$ satisfy the following equality, for $-N \leq n \leq N$

$$Y_{q_{,n}}^{(p,m)} = (L_0)^{q,p}_{n,m}.$$  

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If $\| \cdot \|_2$ denotes the usual 2-norm, we then have the following equality

$$
\| \mathbb{L}^0 x^{(p,m)} - \widehat{\mathbb{L}}_{|m|} x^{(p,m)} \|_2^2 = \sum_{q=1}^{M} \sum_{n=-N}^{N} |(\mathbb{L}^0)_{m,n}^{p,q}|^2.
$$

(23)

Our goal is to prove that $X^{(p,m)}$ is an approximate eigenvector of $\mathbb{L}^0$, with approximate eigenvalue $\widehat{\mathbb{L}}_{|m|}$, for $|m|$ sufficiently large. To this end, we need to find an upper bound for

$$
\| \mathbb{L}^0 x^{(p,m)} - \widehat{\mathbb{L}}_{|m|} x^{(p,m)} \|_2^2.
$$

Indeed, this term measures the error related to the approximation of an eigenvector of $\mathbb{L}^0$ by $X^{(p,m)}$ with approximate eigenvalue $\widehat{\mathbb{L}}_{|m|}$. We first have

$$
\| \mathbb{L}^0 x^{(p,m)} - \widehat{\mathbb{L}}_{|m|} x^{(p,m)} \|_2^2 \leq \left( \sum_{q=1}^{M} \sum_{n=-N}^{N} |(\mathbb{L}^0)_{m,n}^{p,q}| \right)^2.
$$

(24)

Let us now state the following Lemma.

**Lemma 3.** Let $(p,m) \in I$ and $q = 1, \ldots, M$, such that $p \neq q$ and $m \neq 0$. The following inequality holds

$$
\sum_{n=-N}^{N} |(\mathbb{L}^0)_{n,m}^{q,p}| \leq \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
$$

(25)

**Proof.** Let us consider three integers $p, q = 1, \ldots, M$ and $m = -N, \ldots, N$ such that $p \neq q$ and $m \neq 0$. From definition (17) of the coefficients $(\mathbb{L}^0)_{n,m}^{p,q}$, we have

$$
\sum_{n=-N}^{N} |(\mathbb{L}^0)_{n,m}^{q,p}| = \frac{a}{2|m|} \sum_{n=0}^{N} \sum_{n=0}^{N} \frac{|m| + |n| - 1}{|m|!|n|!} \left( \frac{a}{b_{pq}} \right)^{|n|}.
$$

Since $|m|! = |m|(|m| - 1|!|!)$, we can write

$$
\sum_{n=-N}^{N} |(\mathbb{L}^0)_{n,m}^{q,p}| = \frac{a}{2|m|} \sum_{n=0}^{N} \sum_{n=0}^{N} \frac{|m| + |n| - 1}{(|m| - 1)!|n|!} \left( \frac{a}{b_{pq}} \right)^{|n|}.
$$

Then one gets

$$
\sum_{n=-N}^{N} |(\mathbb{L}^0)_{n,m}^{q,p}| = \frac{a}{2|m|} \sum_{n=0}^{N} \sum_{n=0}^{N} \frac{|m| + |n| - 1}{|m|} \left( \frac{a}{b_{pq}} \right)^{|n|},
$$

(26)

where

$$
\left( \frac{|m| + |n| - 1}{|m| - 1} \right) = \frac{(|m| + |n| - 1)!}{(|m| - 1)!|n|!}.
$$
Since \( a < b_{pq} \), the series indexed by \( n \) appearing in the right-hand side of the equality (26) is converging when \( N \) tends towards infinity. More precisely, we have

\[
\lim_{N \to \infty} \sum_{n=0}^{N} \left( \frac{|m| + |n| - 1}{|m| - 1} \right) \left( \frac{a}{b_{pq}} \right)^{|n|} = \left( 1 - \frac{a}{b_{pq}} \right)^{-|m|} = \left( \frac{b_{pq}}{b_{pq} - a} \right)^{|m|}.
\]

Furthermore, since each term of the series is positive, the following inequality is fulfilled for any \( N \in \mathbb{N} \)

\[
\sum_{n=0}^{N} \left( \frac{|m| + |n| - 1}{|m| - 1} \right) \left( \frac{a}{b_{pq}} \right)^{|n|} \leq \left( \frac{b_{pq}}{b_{pq} - a} \right)^{|m|}.
\]

By using the upper bound in (26), we finally obtain

\[
\sum_{n=-N}^{N} |(L^0)_{n,m}| \leq \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
\]

This Lemma allows us to have a fine upper bound of the euclidian norm

\[
\left\| L^0 X^{(p,m)} - \hat{L}_{|m|} X^{(p,m)} \right\|_2
\]

and to show that \( X^{(p,m)} \) can be used as an approximate eigenvector of \( L^0 \). We summarize these results in the following proposition.

**Proposition 2.** Let \( (p, m) \in I \) with \( m \neq 0 \). The vector \( X^{(p,m)} \) defined by relations (21) and (22) is an approximate eigenvector of \( L^0 \) with approximate eigenvalue \( \hat{L}_{|m|} \), in the sense that the relative error between \( L^0 X^{(p,m)} \) and \( \hat{L}_{|m|} X^{(p,m)} \) satisfies the following upper bound

\[
\left\| L^0 X^{(p,m)} - \hat{L}_{|m|} X^{(p,m)} \right\|_2 \leq \sum_{q=1}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
\]

**Proof.** We apply Lemma 3 to the first inequality (24) for the term \( \left\| L^0 X^{(p,m)} - \hat{L}_{|m|} X^{(p,m)} \right\|_2 \) to obtain

\[
\left\| L^0 X^{(p,m)} - \hat{L}_{|m|} X^{(p,m)} \right\|_2 \leq \left( \sum_{q=1}^{M} \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|} \right)^2.
\]

Since all the terms with index \( q \) appearing in the sum are positive, we can take the square-root of the inequality to get

\[
\left\| L^0 X^{(p,m)} - \hat{L}_{|m|} X^{(p,m)} \right\|_2 \leq \frac{a}{2|m|} \sum_{q=1}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
\]
Finally, we use the relation \( \hat{\mu}_{|m|} = \frac{a}{2|m|} \) leading to
\[
\frac{\|L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)}\|_2}{\hat{\mu}_{|m|}} \leq \sum_{q=1}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
\]

This proposition shows that, for \((p, m) \in I\) with \(m \neq 0\) and \(|m|\) sufficiently large, then the vector \(X^{(p,m)}\) is an approximate eigenvector of \(L^0\) with approximate eigenvalue \(\hat{\mu}_{|m|}\). Moreover, the \(2M\) vectors \((X^{(p,m)})_{p=1,\ldots,M}\) and \((X^{(p,-m)})_{p=1,\ldots,M}\) are all approximate eigenvectors associated with the same approximate eigenvalue \(\hat{\mu}_{|m|}\). Hence, for \(|m|\) large enough, the term \(\hat{\mu}_{|m|}\) is an approximate eigenvalue of \(L^0\) with multiplicity \(2M\). The sequence \((\hat{\mu}_{m})_{m \geq 1} = (\frac{a}{2m})_{m \geq 1}\) decays and tends towards 0 when \(m\) tends to infinity. The term \(\hat{\mu}_N\) is then the smallest approximate eigenvalue \((\hat{\mu}_{m})_{m \geq 1}\). Furthermore, \(\hat{\mu}_N\) tends to 0 when \(N\) tends to infinity. This is the reason why we estimate \(\mu_{0\min}\), and next \(\mu_{\min}\), by \(\hat{\mu}_N\) with a multiplicity equal to \(2M\), that is
\[
\mu_{\min} \simeq \mu_{0\min} \simeq \hat{\mu}_N.
\]

Let us remark that \(\hat{\mu}_N\) is also the approximation of the smallest eigenvalue in the framework of single scattering as well as multiple scattering for distant scatterers [6]. This approximation is more accurate as \(N\) is large and that the coupling between the obstacles is weak, that is when the obstacles are not too close. Indeed, the larger the distance \(b_{pq}\) is, the smaller the left hand side of the inequality (27) is. Similarly, the right hand side term of (27) (with \(|m| = N\)) is smaller as \(N\) is larger.

Let us now come back to the example in Figure 3. The parameters were: 30 randomly distributed disks of radius 0.1 in \([0, 4]^2\), \(N = 5\), \(k = 0.1\), \(0.33 \leq b_{pq} \leq 4.59\). The numerical computation of the smallest eigenvalues of \(L\) and \(L^0\) provide \(\mu_{\min} \simeq 0.01 + 10^{-13}i\) and \(\mu_{0\min} \simeq 0.01 + 10^{-13}i\). Our estimate gives \(\hat{\mu}_N = 0.01\), with a small relative error on \(\mu_{\min}\) equal to
\[
100 \frac{|\mu_{\min} - \hat{\mu}_N|}{|\mu_{\min}|} \simeq 0.08\%.
\]

### 3.3 Estimates of the eigenvalue with largest modulus

Unlike the previous case, we cannot construct an approximate eigenvector of \(L^0\) to provide an estimate of the largest eigenvalue \(\mu_{\max}^0\) of \(L^0\). By analyzing the expression (17) of the coefficients of the limit matrix \(L^0\), none of them depends on \(k\), except the coefficients associated with the indices \(m = n = 0\) which have a logarithmic growth with respect to \(k\) (zone 0 in Lemma 2). When the wavenumber \(k\) is small enough, the information related to the largest eigenvalue is a priori contained in these coefficients.

As in [6], we propose to extract the matrix \(L^1\) from \(L^0\) and related to the zero order modes. From relation (17), this \(M \times M\) matrix \(L^1\) is defined, for \(p, q = 1, \ldots, M\), by
\[
(L^1)^{p, q} = (L^0)^{p, q}_0\delta = \begin{cases} -a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + i\frac{\pi a}{2}, & \text{if } p = q, \\ -a \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + i\frac{\pi a}{2}, & \text{if } p \neq q. \end{cases}
\]
Let us denote by $\mu_0$ and $\mu_1$, respectively $\mu_1$ and $\mu_2$, the eigenvalue with largest modulus of $L_0$ and $L_1$. We then estimate the largest eigenvalue of $L_0$ by the one of $L_1$, that is: $\mu_0 \simeq \mu_1$. Next, from (20), we also estimate $\mu$ by $\mu_1$. We compared three approaches to estimate $\mu_1$. Two consist in bounding the spectral radius of $L_1$, and so the modulus of $\mu_1$, by computing the Frobenius norm of $L_1$ or by applying the Gershgorin’s discs theorem to $L_1$. In both cases, the result was less accurate than for the third approach. Moreover, the expression of the estimate obtained by this last method is simpler. This is the reason why we only present this approach here.

The principle of our approach is to obtain a mean distance $d$ related to the inter center distances $b_{pq}$. Let us introduce $L_1^{eqv}$ as the matrix of size $M \times M$ defined by

$$
\forall p, q = 1, \ldots, M, \quad (L_1^{eqv})^{p,q} = \begin{cases} 
-a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{\pi a}{2}, & \text{for } p = q, \\
-a \left[ \ln \left( \frac{kd}{2} \right) + \gamma \right] + \frac{\pi a}{2}, & \text{for } p \neq q, 
\end{cases}
$$

where $d > 0$ is an “equivalent” distance related to the distances $b_{pq}$ (the coefficients $(L_1^{eqv})^{p,q}$ are obtained by replacing $b_{pq}$ by $d$ in (29)). We suppose that $d$ verifies the inequality

$$
d \geq \min_{1 \leq p \neq q \leq M} b_{pq}. \tag{30}
$$

Let us remark that the matrix $L_1^{eqv}$ is defined through two parameters by

$$
L_1^{eqv} = \begin{pmatrix}
l_0 & l_1 & l_1 & \ldots & l_1 \\
l_1 & l_0 & l_1 & \ldots & l_1 \\
l_1 & l_1 & l_0 & \ldots & l_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
l_1 & \ldots & l_1 & l_1 & l_0
\end{pmatrix},
$$

with

$$
\begin{cases}
l_0 = (L_1^{eqv})^{0,0} = -a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{\pi a}{2}, \\
l_1 = (L_1^{eqv})^{1,1} = -a \left[ \ln \left( \frac{kd}{2} \right) + \gamma \right] + \frac{\pi a}{2}.
\end{cases} \tag{31}
$$

The main property of this matrix is that we explicitly know its eigenvalues, and most particularly its largest one, accordingly to the next Lemma.

**Lemma 4.** The eigenvalues of $L_1^{eqv}$ are given by

- $\mu_1 = l_0 + (M - 1)l_1 = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i \frac{M\pi a}{2}$, with multiplicity 1,

- $\mu_2 = l_0 - l_1 = a \ln \left( \frac{d}{a} \right)$, with multiplicity $(M - 1)$.

Moreover, for $kd < 1$, we have the following inequality

$$
|\mu_1| \geq |\mu_2|. \tag{32}
$$
Proof. The characteristic polynomial of $L_{\text{equiv}}^1$ is given by: $P(X) = \det(L_{\text{equiv}}^1 - XI)$, where $I$ is the $M \times M$ identity matrix,

\[
\det(L_{\text{equiv}}^1 - XI) = \det \begin{pmatrix}
    l_0 - X & l_1 & l_1 & \ldots & l_1 \\
    l_1 & l_0 - X & l_1 & \ldots & l_1 \\
    l_1 & l_1 & l_0 - X & \ldots & l_1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    l_1 & \ldots & l_1 & l_1 & l_0 - X
\end{pmatrix}.
\]

We substract the first line to lines 2, 3, ..., $M$ to get

\[
\Det(L_{\text{equiv}}^1 - XI) = \Det \begin{pmatrix}
    l_0 - X & l_1 & l_1 & \ldots & l_1 \\
    l_1 - l_0 + X & l_0 - X - l_1 & 0 & \ldots & 0 \\
    l_1 - l_0 + X & 0 & l_0 - X - l_1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    l_1 - l_0 + X & \ldots & 0 & 0 & l_0 - X - l_1
\end{pmatrix}.
\]

Next, we add to the first column the other $M - 1$ columns

\[
\Det(L_{\text{equiv}}^1 - XI) = \Det \begin{pmatrix}
    l_0 - X + (M - 1)l_1 & l_1 & l_1 & \ldots & l_1 \\
    0 & l_0 - X - l_1 & 0 & \ldots & 0 \\
    0 & 0 & l_0 - X - l_1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & 0 & l_0 - X - l_1
\end{pmatrix}.
\]

Hence, we obtain easily the expression

\[
\Det(L_{\text{equiv}}^1 - XI) = (-1)^M [X - (l_0 + (M - 1)l_1)] [X - (l_0 - l_1)]^{M-1}.
\]

We then deduce the eigenvalues of $L_{\text{equiv}}^1$

\[
\mu_1^1 = l_0 + (M - 1)l_1 = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2},
\]

with multiplicity 1, and

\[
\mu_2^1 = l_0 - l_1 = a \ln \left( \frac{d}{a} \right),
\]

with multiplicity $M - 1$.

Let us now state the inequality (32). We first remark that the eigenvalue $\mu_2^1$ is real. From relation (31) and since $d > a$, $\mu_2^1$ is positive. It remains to prove that the real part of $\mu_1^1$ is larger than $\mu_2^1$ to obtain inequality (33). Let us begin by noticing that, since the eigenvalue $\mu_2^1$ is real, it can be written $\mu_2^1 = \Re(l_0) - \Re(l_1)$. We next compare the real part of $\mu_1^1$ with $\mu_2^1$

\[
\Re(\mu_1^1) - \Re(\mu_2^1) = (M - 2)\Re(l_1).
\]

Since $M - 2 \geq 0$ (because we have at least two scatterers), $\Re(\mu_1^1) - \Re(\mu_2^1)$ and $\Re(l_1)$ have the same signs. We therefore try to prove that $\Re(l_1)$ is positive. Let us give the expression of the real part of $l_1$

\[
\Re(l_1) = -a \left[ \ln \left( \frac{kd}{2} \right) - \gamma \right].
\]
Let us recall that: 
\[- \ln(0.5) \simeq 0.69 > \gamma \simeq 0.58.\]
We assume that we have \( kd < 1 \) (since \( k \) tends towards 0). Since \( \ln \) is an increasing function, we get
\[- \ln\left(\frac{kd}{2}\right) \geq \gamma > 0,\]
that is
\[\Re(l_1) = a\left(- \ln\left(\frac{kd}{2}\right) - \gamma\right) \geq 0.\]
This implies that, from relation (33), we have: \( \Re(\mu_1) = - \Re(\mu_2) \geq 0 \), and next \( |\mu_1| \geq |\mu_2| \).

This Lemma shows that the eigenvalue \( \mu_{\text{max}}^{1,\text{equiv}} \) of \( L_{\text{equiv}}^{1} \) with largest modulus has a multiplicity equal to 1 and can be expressed as
\[
\mu_{\text{max}}^{1,\text{equiv}} = l_0 + (M - 1)l_1 = -a\left[\ln\left(\frac{ka}{2}\right) + (M - 1)\ln\left(\frac{kd}{2}\right)\right] - aM\gamma + i\frac{M\pi a}{2}. \tag{34}
\]

We now estimate \( \mu_{\text{max}} \) by \( \mu_{\text{max}}^{1,\text{equiv}} \), that is
\[
\mu_{\text{max}} \simeq -a\left[\ln\left(\frac{ka}{2}\right) + (M - 1)\ln\left(\frac{kd}{2}\right)\right] - aM\gamma + i\frac{M\pi a}{2}.
\]

We propose to choose \( d \) as
\[
d = \frac{b_{\text{min}} + b_{\text{max}}}{2}.
\]
The term \( b_{\text{min}}, \) respectively \( b_{\text{max}}, \) represents the smallest, respectively largest, possible distance \( b_{pq} \) between the centers of two obstacles. When the obstacles are contained in a rectangular box of sides \( \ell \) and \( L \), we fix \( b_{\text{min}} \) and \( b_{\text{max}} \) as: \( b_{\text{min}} = 2a \) and \( b_{\text{max}} = \sqrt{\ell^2 + L^2} - 2a \). Finally, \( d \) is defined as the average of \( b_{\text{min}} \) and \( b_{\text{max}} \)
\[
d = \frac{b_{\text{min}} + b_{\text{max}}}{2} = \frac{\sqrt{\ell^2 + L^2}}{2}. \tag{35}
\]

Let us remark that \( d \) can also be seen as the half diagonal of the box with sidelenghts \( \ell \) and \( L \). Let us come back again to the example presented in Figure 3. The numerical computation of the largest eigenvalue \( \mu_{\text{max}} \) of \( L \) was \( \mu_{\text{max}} \simeq 5.39 + 4.62i \), with \( |\mu_{\text{max}}| \simeq 7.1 \). By using relation (35) for \( d \), one gets \( d = 2\sqrt{2} \). The proposed formula (34) then gives \( \mu_{\text{max}}^{1,\text{equiv}} \simeq 4.47 + 4.71i \), with \( |\mu_{\text{max}}^{1,\text{equiv}}| \simeq 6.5 \). The relative error when estimating \( \mu_{\text{max}} \) by \( \mu_{\text{max}}^{1,\text{equiv}} \) is then equal to
\[
100\left|\frac{\mu_{\text{max}} - \mu_{\text{max}}^{1,\text{equiv}}}{\mu_{\text{max}}}ight| = 13.04\%, \tag{36}
\]
which means that our approach is consistent. We launched 100 tests respecting the same parameters set \( (k = 0.1, N = 5, 30 \text{ randomly distributed disks of radius } 0.1 \text{ in } [0,4]^2 \text{ with } b_{pq} \geq 0.1) \). The results are reported on Figure 4. We observe on Figures 4(a) and 4(b) that the error essentially affects the real part of \( \mu_{\text{max}} \), the imaginary part estimate being acceptable. The average relative error on the modulus of \( \mu_{\text{max}} \) for these 100 realizations is about 15.6\%.

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Figure 4: Comparisons between $\mu_{\text{max}}$ and its estimate $\mu_{\text{max}}^{\text{equiv}}$ for 100 configurations of $M = 30$ obstacles with radius $a = 0.1$ randomly placed in $[0,4] \times [0,4]$, with $k = 0.1$, $N = 5$ and $b \geq 0.1$.

### 3.4 Condition number estimate

Since we have an estimate of both the smallest eigenvalue $\mu_{\text{min}}$ (see (28)) and largest eigenvalue $\mu_{\text{max}}$ (see (34)) of $L$, we obtain the following approximation of the condition number

$$\text{cond}(L) \simeq \text{cond}_{\text{app}}(L) = 2N \left| - \left[ \ln \left( \frac{ka}{2} \right) + \ln \left( \frac{k'd'}{2} \right) \right] - M\gamma + i\frac{M\pi}{2} \right|. \quad (37)$$

Let us consider again the example of Figure 3. Then, the numerical condition number of $L$ is: $\text{cond}(L) \simeq 713$ while the estimate (37) yields $\text{cond}_{\text{app}}(L) \simeq 650$, leading to a relative error

$$\frac{100\left| \text{cond}(L) - \text{cond}_{\text{app}}(L) \right|}{\left| \text{cond}(L) \right|} \simeq 9\%.$$ \quad (38)

Essentially, this error is related to the estimate on the largest eigenvalue. We report on Figure 5 the results for 100 launches. This gives a mean relative error equal to 11\%, which is satisfactory.
4 Connections with the boundary element approximation and extension to other geometries

We approximate the variational formulation (2) by a linear boundary element method. For more details we refer to [6] where a similar approach is developed for distant obstacles. Here, we only provide the notations used later. The parameter $h$ designates the smallest element size of the $N_{\text{tot},h}$ segments required for the discretization by polygonal curves $\Gamma_h$ of $\Gamma$ (the union of all the scatterers).

The boundary element space $V_h$ is the subspace of $L^2(\Gamma)$ defined by

$$V_h := \{ \rho_h \in C^0(\Gamma_h) \text{ such that } \rho_h|_{K_{p,j}} \in P_1, \text{ with } 1 \leq p \leq M \text{ and } 1 \leq j \leq N_h \}.$$  

The discrete boundary element formulation of eq. (2) then writes

$$\begin{cases}
\text{Find } (\rho_h, \mu_h) \in C^{N_{\text{tot},h}} \times C \\
[L_h] \rho_h = -\mu_h [M_h] \rho_h
\end{cases}$$

where $[L_h] \in M_{N_{\text{tot},h},N_{\text{tot},h}}(C)$ is the single-layer matrix and $[M_h] \in M_{N_{\text{tot},h},N_{\text{tot},h}}(C)$ the global mass matrix for linear finite element, $\rho_h \in C^{N_{\text{tot},h}}$ being the eigenvector and $\mu_h$ the numerical eigenvalue. Finally, we denote by $\mu_{\text{min}}^h$, respectively $\mu_{\text{max}}^h$, the smallest, respectively largest, eigenvalue of the matrix $[M_h]^{-1} [L_h]$.

4.1 The circular cylinder case

We relate our spectral analysis to the boundary element approximation. As in [6], we begin by considering the case of circular cylinders and then formally extend the results to rectangular and elliptical shaped objects. We consider $M$ disks $\Omega_p^-$ with the same radius $a$ and we uniformly mesh each circle $\Gamma_p$ by fixing the meshsize to $h$. Following [6], we formally substitute $N$ by $\pi a h^{-1} - 1/2$.
in the estimate (28), respectively (34), of \( \mu_{\min} \), respectively \( \mu_{\max} \). When \( k \) tends towards 0, we therefore obtain

\[
\begin{align*}
\mu_{\min} & \simeq \mu_{\min}^{\text{app}}(a, h), \\
\mu_{\max} & \simeq \mu_{\max}^{\text{app}}(a, k),
\end{align*}
\]

with

\[
\begin{align*}
\mu_{\min}^{\text{app}}(a, h) & = \frac{a}{2\pi ah^{-1} - 1}, \\
\mu_{\max}^{\text{app}}(a, k) & = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi}{2},
\end{align*}
\]

(39)

where \( d \) represents a mean distance between obstacles. When the obstacles are contained in the rectangular box \([0, \ell] \times [0, L]\), we use the previous expression (35)

\[
d = \sqrt{\ell^2 + L^2}.
\]

In addition, the condition number of the matrix \([M_h]^{-1} [L_h]\) is approximated, when \( k \) tends towards zero, by

\[
\text{cond}(k, a, \Gamma_h) := \text{cond}_2([M_h]^{-1} [L_h]) \simeq \text{cond}^{\text{app}}(k, a, h),
\]

with

\[
\text{cond}^{\text{app}}(k, a, h) = a(2\pi ah^{-1} - 1) \left| - \ln \left( \frac{ka}{2} \right) - (M - 1) \ln \left( \frac{kd}{2} \right) - aM\gamma + i\frac{M\pi}{2} \right|.
\]

(40)

We propose to numerically validate the above approximations for the proceeding example: \( M = 30 \) small disks of radius \( a = 0.1 \) are placed inside the box \([0, 4]\)^2. The smallest inter centers distance \( b \) is equal to 0.3(= 3a) and \( k = 0.1 \). Moreover, each obstacle is meshed with \( N_h = 50 \) elements. We numerically compute the eigenvalues and the condition number of \([M_h]^{-1} [L_h]\) for 100 configurations as well as their corresponding estimates. Let us begin by comparing the numerical \( \mu_{\min}^h \) and \( \mu_{\max}^h \) and estimated eigenvalues on Figure 6. The two first figures 6(a) and 6(b) show that the relative error on the smallest eigenvalue \( \mu_{\min}^h \) is about 2.4%, which is very satisfactory and similar to the distant obstacles case. Concerning the largest eigenvalue, the relative error is about 15%, the main deterioration being on the real part of \( \mu_{\max}^h \). By comparison, we get a similar error of about 13% with the spectral method. We compare now on Figure 7 the variations of the condition number of \([M_h]^{-1} [L_h]\) with its estimate (40). We obtain a mean relative error of 13% which is about the same as for the spectral method (11%).

### 4.2 Extension to other geometries

We now formally adapt the estimates (39) for the eigenvalues \( \mu_{\min}^h \) and \( \mu_{\max}^h \) first to elliptical and then to rectangular cylinders. We proceed as we did in [6] for dilute media: we formally replace the radius \( a \) and the meshsize \( h \) in the estimates (39) of the eigenvalues \( \mu_{\min}^h \) and \( \mu_{\max}^h \) by respectively an equivalent radius \( a_{\text{eqv}} \) and an equivalent step \( h_{\text{eqv}} \). For an ellipse with semi-axis \( a_{x_1} \) along the
Figure 6: Comparison of the estimates (39) of the modulus of the smallest and largest eigenvalues of the matrix $[M_h]^{-1}[L_h]$. The obstacles are small disks of radius $a = 0.1$ discretized by using $N_h = 50$ segments. For each of the 100 configurations, we randomly place $M = 30$ disks in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3 (= 3a)$. 
direction $x_1$ and $a_{x_2}$ along $x_2$, we proposed to choose an equivalent mesh parameter $h_{\text{equiv}}$ equal to the smallest discretization meshsize. Moreover the three equivalent radii were considered [6]

$$a_{\text{equiv}}^1 = \frac{x_1 + a_{x_2}}{2}, \quad a_{\text{equiv}}^2 = \frac{2a_x, a_{x_2}}{a_{x_1} + a_{x_2}} \quad \text{et} \quad a_{\text{equiv}}^3 = \sqrt{a_{x_1}^2 + a_{x_2}^2}. \sqrt{2}.$$

We propose to validate these approximations for small ellipses with semi-axes $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$. We keep the same parameters as before (30 obstacles randomly placed in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3(= 3a)$.

Figure 7: Comparison between the condition number of the matrix $[M_h]^{-1}[L_h]$ and its estimate (40). The obstacles are small disks of radius $a = 0.1$ discretized by using $N_h = 50$ segments. For each of the 100 configurations, $M = 30$ disks are randomly placed in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3(= 3a)$.

We begin by comparing the estimates of $\mu_{\text{min}}^h$ and $\mu_{\text{max}}^h$ on Figure 8. Like for the single scattering situation [6], choosing an equivalent radius has almost no effect on the estimate of $\mu_{\text{min}}^h$. Furthermore, the relative error on $|\mu_{\text{min}}^h|$ is of the order of 18%, for each radius. This important error can be explained by the fact that $\mu_{\text{min}}^h$ is strongly mesh dependent, and most particularly relatively to the smallest mesh size (strong curvature effects). Hence, the estimate of the smallest eigenvalue can clearly degenerate. Concerning the eigenvalue $\mu_{\text{max}}^h$ with largest modulus, it is directly impacted by the choice of the equivalent radius. Indeed, the four Figures 6(c)-6(f) show that $a_{\text{equiv}}^3$ leads to a better approximation of $\mu_{\text{max}}^h$ than $a_{\text{equiv}}^1$. More precisely, the relative error on the modulus of $\mu_{\text{max}}^h$ is about 14% for $a_{\text{equiv}}^3$ compared with 22% for $a_{\text{equiv}}^1$. When the obstacles are distant, we observed an opposite behaviour. Finally, we compare on Figure 9 the condition number of the matrix $[M_h]^{-1}[L_h]$ with its estimate. We have only reported the results related to $a_{\text{equiv}}^3$ since it leads to the best approximation of $\mu_{\text{max}}^h$. The relative error on the condition number is about 9%
which is very satisfying.

We end this numerical study by considering rectangular obstacles denoting by $a_{x_1}$ and $a_{x_2}$ its half side lengths. We take the equivalent radius $a_{eqv}^4$ given by [6]

$$a_{eqv}^4 = \frac{(1 + \sqrt{2}) \sqrt{a_{x_1}^2 + a_{x_2}^2}}{2 \sqrt{2}}.$$ 

Moreover, the equivalent step $h_{eqv}$ is always chosen equal to the smallest discretization meshsize

$$h_{eqv} = \min_{1 \leq p \leq M} \min_{1 \leq j \leq N_{h,p}} h_{p,j}.$$ 

We provide a numerical example considering the previous parameters: $M = 30$ randomly placed rectangular cylinders with half side lengths $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$ in $[0, 4]^2$. Moreover, we fix $k = 0.1$ and the smallest distance $b$ between two obstacles is equal to 0.15. Numerically, we compute the eigenvalues of the matrix $[M_h]^{-1} [L_h]$, its condition number as well as their respective estimates for 100 configurations. We begin by comparing on Figure 10 the estimates of the eigenvalues with smallest, respectively largest, modulus $\mu_h^{min}$, respectively $\mu_h^{max}$. The average error on the modulus of $\mu_h^{min}$ is 2%, compared with 18% in the elliptical case. This can be explained by the property that, in the rectangular case, the mesh is non uniform but the mesh size is constant on each of the four rectangle sides (unlike the ellipse). Concerning the eigenvalue with largest modulus $\mu_h^{max}$, the relative error is about 13%, which is of the same order as for disks and ellipses. Finally, the condition number estimates is satisfying since, from Figure 11, the average relative error is about 14% like for disks.
Figure 8: Comparison of the estimates (39) of the smallest and largest eigenvalues of $[M_h]^{-1}[L_h]$. The obstacles are small ellipses with semi axis $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$, obtained for a discretization with $N_h = 50$ segments. For each of the 100 configurations, $M = 30$ ellipses are randomly distributed in $[0, 4]^2$, setting $k = 0.1$ and $b \geq 0.3(= 3a_{x_1})$. 
Figure 9: Comparison between the condition number of the matrix $[M_h]^{-1}[L_h]$ and its estimate. The obstacles are ellipses with semi-axis $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$, discretized with $N_h = 50$ segments. For each of the 100 configurations, $M = 30$ ellipses are randomly distributed in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3(= 3a_{x_1})$. 

(a) Comparison between the exact and approximate condition numbers

(b) Relative error on the condition number
Figure 10: Comparison of the estimates of the modulus of the smallest and largest eigenvalues of the matrix $[M_h]^{-1}[L_h]$ for rectangular cylinders with semi-axis $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$. Each rectangle is discretized with $N_h = 48$ segments (12 by edge). For each of the 100 configurations, $M = 30$ rectangular cylinders are randomly placed in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3(= 3a_{x_1})$. 

(a) Comparison between the modulus of the smallest eigenvalue and its estimate

(b) Relative error on the modulus of $\mu_{\text{min}}$

(c) Comparison between the real part of the largest eigenvalue and its estimate

(d) Comparison between the imaginary part of the largest eigenvalue and its estimate

(e) Comparison between the modulus of the largest eigenvalue and its estimate

(f) Relative error on the modulus of $\mu_{\text{max}}$
Figure 11: Comparison between the condition number of the matrix $[M_h]^{-1}[L_h]$ and its estimate. The obstacles are rectangles with half side lengths $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$, each being discretized with $N_h = 48$ elements (12 by edge). For each of the 100 configurations, $M = 30$ rectangles are randomly distributed in $[0,4]^2$, setting $k = 0.1$ and $b \geq 0.3(= 3a_{x_1})$.

5 Conclusion

In this second and last part, we developed and validated low-frequency spectral and condition number estimates of the single-layer integral operator for dense multiple scattering media. They have been formally extended to circular, elliptical and rectangular shaped obstacles when a linear boundary element is considered. These estimates provide explicit dependence of the eigenvalues with respect to the different problem parameters.

These studies open different directions which should complete this work. First, spectral estimates related to the Brakhage-Werner integral equation [4, 5, 9] and Combined Field Integral Equation [4, 5, 19] can be expected in similar situations since they involve the four basic integral operators: the single- and double-layer potentials as well as their normal derivatives. A difficult situation that is not studied here is the case where the distance between the obstacles is of the order of the characteristic size of the scatterers. We did not succeed yet in deriving similar estimates. One of the main difficulties is that an asymptotic regime is not available. Finally, considering the medium/high frequency is of interest but leads to complex spectral distributions which require further studies. Indeed, many modes couple leading to complex spectral features.

References


